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FUZZY LIMITS OF FUZZY FUNCTIONS

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ABSTRACT This work aims to investigate the theory of fuzzy limits of fuzzy functions in light of Altai's principle, and by applying the Representation Theorem (Resolution Principle) to run fuzzy arithmetics. The novelty underlying this theory is that we can prove the convergence of a fuzzy function to its fuzzy limit through proving the convergence of its α-cuts' boundaries to their limits for the membership degree $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$.

Keywords: fuzzy limit of fuzzy function, two-sided fuzzy limits, one-sided fuzzy limits, fuzzy limit at infinity, infinity fuzzy limit.

1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy sets was introduced by Zadeh as an extension of the classical notion of sets [12]. The concept of the fuzzy metric space was defined using the continuous t-norms concept by Kramosil and Michalek [7]. The fuzzy metric spaces offer numerous applications in quantum physics, particularly, in connection to both string and $\varepsilon^{(\infty)}$ theories [11]. Matloka considered bounded and convergent sequences of fuzzy numbers, and investigated their properties [9]. Sequences of fuzzy numbers were also discussed by Nanda [10], Kwon [8], and Esi [6]. Burgin introduced the theory of the fuzzy limits of functions based on the theory of the fuzzy limits of sequences. He studied and developed the construction of fuzzy limits of functions in a similar way to the one of the fuzzy limits of sequences, based on the concept of the r-limit of the function f (see [3-4]). Altai established a novel principle to define fuzzy metric spaces through the families of fuzzy real numbers $\overline{\overline{\mathbb{R}}}$, fuzzy integer numbers $\overline{\overline{\mathbb{Z}}}$, fuzzy rational numbers $\overline{\overline{\mathbb{Q}}}$ and fuzzy irrational numbers $\overline{\overline{\mathbb{Q}'}}$, where $\overline{\overline{\mathbb{R}}} = \overline{\overline{\mathbb{Z}}} \cup \overline{\overline{\mathbb{Q}}} \cup \overline{\overline{\mathbb{Q}'}}$. His principle states that if r is a real number, then it is replaced by a fuzzy real number $\overline{r} \in \overline{\overline{\mathbb{R}}}$, such that if r is a rational or an irrational number, then $\overline{r}$ becomes a fuzzy rational number (triangular fuzzy number in $\overline{\overline{\mathbb{Q}}}$) or a fuzzy irrational number (fuzzy rational number in $\overline{\overline{\mathbb{Q}'}}$) respectively. Additionally, if r is an integer number, then $\overline{r}$ becomes a singleton fuzzy number (fuzzy integer number in $\overline{\overline{\mathbb{Z}}}$) (see [1] for details). This involves adopting the Representation Theorem (Resolution Principle) to run the arithmetic operations acting on the α-cuts of fuzzy numbers [5]. Moreover, Altai defined the fuzzy limit of the convergent fuzzy sequence depending on the same principle [2].

This study aims to introduce the theory of fuzzy limits of fuzzy functions depending on Altai's principle, because it is convenient in the study of fuzzy arithmetic [5]. Moreover, since for all $\alpha \in (0,1]$, the α-cut of a fuzzy number is a closed, convex and compact subset of $\mathbb{R}$, the existence, uniqueness and all other basic properties of the fuzzy limit of a fuzzy function can be studied, depending on the limits of all its α-cuts' boundaries.

For the convenience of the reader, we state below the Representation Theorem and the Resolution Principle as obtained in [5].

Representation theorem [5]. Let $\mu_A(x)$ be the membership function with a fuzzy set A in X, and let A_α be the α-cuts of A and χ_{A_α} be the characteristic to serve the purpose of the fuzzy set $A_\alpha, \alpha \in (0,1]$. Then

$$\mu_A(x) = \sup_{\alpha\in(0,1]} \left(\alpha \wedge \chi_{A_\alpha}(x)\right), x \in X.$$

Resolution principle [5]. Let A be a fuzzy set in X and $\alpha A_\alpha, \alpha \in (0,1]$ be a special fuzzy set whose membership function is

$$\mu_{\alpha A_\alpha}(x) = \left(\alpha \wedge \chi_{A_\alpha}(x)\right), x \in X.$$

Also, let

$$\Lambda_A = \{\alpha: \mu_A(x) = \alpha \quad \text{for} \quad \text{some} \quad x \in X\}$$

be the level set of A. Then A can be formulated as

$$A = \bigcup_{\alpha\in\Lambda_A} (\alpha A_\alpha),$$

where $\bigcup$ denotes the standard fuzzy union.

The essence of the Representation Theorem of fuzzy sets is that a fuzzy set A in X can be represented as a union of its αA_α sets, $\alpha \in (0,1]$. The essence of the Resolution Principle is that a fuzzy set A can be decomposed into fuzzy sets $\alpha A_\alpha, \alpha \in (0,1]$. Thus, this theorem and the Resolution Principle are the same coin with two sides, as both of them essentially tell that a fuzzy set A in X can always be expressed in terms of its α-cuts, without explicitly resorting to its membership function $\mu_A(x)$ (see[5]).

2. TWO-SIDED FUZZY LIMITS

The definition of the fuzzy limit of a fuzzy function are introduced and its basic properties are considered hereafter.

Definition 2.1 [1]. Let $\overline{\overline{X}}$ be a family of fuzzy numbers. A fuzzy function $d: \overline{\overline{X}} \times \overline{\overline{X}} \to \overline{\overline{\mathbb{R}}}$ is called a fuzzy metric on $\overline{\overline{X}}$ if

1. $d(\bar{x}, \bar{y}) \geq 0$, for all $\bar{x}, \bar{y} \in \overline{\overline{X}}$; i.e, for $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$,

$$\mathrm{d}_1\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right) = \min\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \geq 0$$

and

$$\mathrm{d}_2\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right) = \max\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \geq 0.$$

2. $d(\bar{x},\bar{y}) = 0$ iff $\bar{x} = \bar{y}$; i.e, for $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$,

$$\mathrm{d}_1\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right) = \min\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} = 0$$

and

$$\mathrm{d}_2\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right) = \max\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} = 0.$$

3. $d(\bar{x},\bar{y}) = d(\bar{y},\bar{x})$, for all $\bar{x},\bar{y} \in \overline{\overline{X}}$; i.e, for $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$,

$$\begin{aligned}\mathrm{d}_1\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right) &= \min\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \\ &= \min\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \\ &= \mathrm{d}_1\left(\left(y_{1,\alpha}, y_{2,\alpha}\right), \left(x_{1,\alpha}, x_{2,\alpha}\right)\right),\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}_2\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right) &= \max\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \\ &= \max\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \\ &= \mathrm{d}_2\left(\left(y_{1,\alpha}, y_{2,\alpha}\right), \left(x_{1,\alpha}, x_{2,\alpha}\right)\right).\end{aligned}$$

4. $d(\bar{x},\bar{y}) \leq d(\bar{x},\bar{z}) + d(\bar{z},\bar{y})$, for all $\bar{x},\bar{y},\bar{z} \in \overline{\overline{X}}$; i.e, for $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$,

$$\begin{aligned}\mathrm{d}_1\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right) &= \min\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \\ &\leq \min\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(z_{1,\alpha}, z_{2,\alpha}\right)\right)\right\} \\ &\quad + \min\left\{d\left(\left(z_{1,\alpha}, z_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \\ &= \mathrm{d}_1\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(z_{1,\alpha}, z_{2,\alpha}\right)\right) \\ &\quad + \mathrm{d}_1\left(\left(z_{1,\alpha}, z_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right),\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}_2\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right) &= \max\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \\ &\leq \max\left\{d\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(z_{1,\alpha}, z_{2,\alpha}\right)\right)\right\} \\ &\quad + \max\left\{d\left(\left(z_{1,\alpha}, z_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right)\right\} \\ &= \mathrm{d}_2\left(\left(x_{1,\alpha}, x_{2,\alpha}\right), \left(z_{1,\alpha}, z_{2,\alpha}\right)\right) \\ &\quad + \mathrm{d}_2\left(\left(z_{1,\alpha}, z_{2,\alpha}\right), \left(y_{1,\alpha}, y_{2,\alpha}\right)\right).\end{aligned}$$

The pair $\left(\overline{\overline{X}}, d\right)$ is called a fuzzy metric space.

Theorem 2.2. Let $\left(\overline{\overline{X}}, \rho\right)$ and $\left(\overline{\overline{Y}}, d\right)$ be fuzzy metric spaces. Suppose that $f\colon \overline{\overline{E}} \subset \overline{\overline{X}} \longrightarrow \overline{\overline{Y}}, \bar{p}$ is a fuzzy limit point of $\overline{\overline{E}}$ and $\bar{L} \in \overline{\overline{Y}}$. If, for $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, the α-cuts' boundaries of $f(\bar{x})$ converge to the α-cuts' boundaries of $\bar{L}$ as the α-cuts' boundaries of $\bar{x}$ approach the α-cuts' boundaries of $\bar{p}$, then $f(\bar{x})$ converges partially to $\bar{L}$ as $\bar{x} \longrightarrow \bar{p}$.

Proof. For $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, let $[f_1(x_{1,\alpha}, x_{2,\alpha}), f_2(x_{1,\alpha}, x_{2,\alpha})]$ and $[L_{1,\alpha}, L_{2,\alpha}]$ be α-cuts of $f(\overline{x})$ and $\overline{L}$, respectively, such that, for $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$, such that

$$0 < \rho_1\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right) < \delta_1 \text{ implies } d_1(f_1(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}) < \varepsilon,$$
$$0 < \rho_2\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right) < \delta_2 \text{ implies } d_2(f_2(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}) < \varepsilon,$$

where

$$d_1(f_1(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}) = \min\{d(f_1(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}): i = 1,2\},$$
$$d_2(f_2(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}) = \max\{d(f_2(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}): i = 1,2\},$$
$$\rho_1\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right) = \min\left\{\rho\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right)\right\},$$
$$\rho_2\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right) = \max\left\{\rho\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right)\right\}.$$

If $f_*(x_{1,\alpha}, x_{2,\alpha}) \in \left(f_1(x_{1,\alpha}, x_{2,\alpha}), f_2(x_{1,\alpha}, x_{2,\alpha})\right)$, then

$$0 < \rho_1\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right) < \rho_*\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right) < \rho_2\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right) < \delta_*,$$

which implies

$$d_1(f_1(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}) < d_*(f_*(x_{1,\alpha}, x_{2,\alpha}), L_{*,\alpha}) < d_2(f_1(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}) < \varepsilon,$$

by the squeeze theorem for functions, where $\delta_* = \min\{\delta_1, \delta_2\}$ and $L_{*,\alpha} \in (L_{1,\alpha}, L_{2,\alpha})$. That is, the α-cut $[f_1(x_{1,\alpha}, x_{2,\alpha}), f_2(x_{1,\alpha}, x_{2,\alpha})]$ of $f(\overline{x})$ converges to the α-cut $[L_{1,\alpha}, L_{2,\alpha}]$ of $\overline{L}$ as the α-cut $[x_{1,\alpha}, x_{2,\alpha}]$ of $\overline{x}$ approaches the α-cut $[p_{1,\alpha}, p_{2,\alpha}]$ of $\overline{p}$ for $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$. By the Resolution Principle, we obtain the result.

Corollary 2.3. Let $(\overline{\overline{X}}, \rho)$ and $(\overline{\overline{Y}}, d)$ be fuzzy metric spaces. Suppose that $f: \overline{\overline{E}} \subset \overline{\overline{X}} \longrightarrow \overline{\overline{Y}}, \overline{p}$ is a fuzzy limit point of $\overline{\overline{E}}$ and $\overline{L} \in \overline{\overline{Y}}$. If, for all $\alpha \in (0,1]$, the α-cuts' boundaries of $f(\overline{x})$ converge to the α-cuts' boundaries of $\overline{L}$ as the α-cuts' boundaries of $\overline{x}$ approach the α-cuts' boundaries of $\overline{p}$, then $f(\overline{x})$ converges totally to $\overline{L}$ as $\overline{x} \longrightarrow \overline{p}$.

Theorem 2.4. Let $(\overline{\overline{X}}, \rho)$ and $(\overline{\overline{Y}}, d)$ be fuzzy metric spaces. Suppose that $f: \overline{\overline{E}} \subset \overline{\overline{X}} \longrightarrow \overline{\overline{Y}}$ and $\overline{p}$ is a fuzzy limit point of $\overline{\overline{E}}$. Then $f(\overline{x})$ converges partially to $\overline{L} \in \overline{\overline{Y}}$ as $\overline{x} \longrightarrow \overline{p}$ if and only if (iff) for $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \left\|\left(\rho_1\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right), \rho_2\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right)\right)\right\| < \delta,$$

implies (2.1)

$$\left\|\left(d_1\left(f\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right),d_2\left(f\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)\right)\right\|<\varepsilon.$$

Proof. Suppose that $f(\overline{x})$ converges to $\overline{L}\in\overline{\overline{Y}}$ as $\overline{x}\longrightarrow\overline{p}$. By Theorem 2.2, for $0<\alpha_o<\alpha_1\leq\alpha\leq 1$, for all $\varepsilon>0$, there exist $\delta_1,\delta_2>0$ such that

$$0<\rho_1\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)<\frac{\delta_1}{\sqrt{2}}\quad\text{implies}\quad d_1\left(f_1\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)<\frac{\varepsilon}{\sqrt{2}},$$
$$0<\rho_2\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)<\frac{\delta_2}{\sqrt{2}}\quad\text{implies}\quad d_2\left(f_2\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)<\frac{\varepsilon}{\sqrt{2}}.$$

Then

$$0<\left\|\left(\rho_1\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right),\rho_2\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)\right)\right\|=$$
$$\left(\left(\rho_1\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)\right)^2+\left(\rho_2\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)\right)^2\right)^{\frac{1}{2}}<\delta$$

implies

$$\left\|\left(d_1\left(f_1\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right),d_2\left(f_2\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)\right)\right\|=$$
$$\left(\left(d_1\left(f_1\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)\right)^2+\left(d_2\left(f_2\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)\right)^2\right)^{\frac{1}{2}}<\varepsilon$$

where $\delta=\min\{\delta_1,\delta_2\}$.

For the other direction, suppose, for $0<\alpha_o<\alpha_1\leq\alpha\leq 1$, (2.1) is given. Since

$$\rho_1\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)$$
$$\leq\left\|\left(\rho_1\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right),\rho_2\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)\right)\right\|,$$
$$\rho_2\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)$$
$$\leq\left\|\left(\rho_1\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right),\rho_2\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)\right)\right\|$$

and

$$d_1\left(f_1\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)\leq\left\|\left(d_1\left(f_1\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right),d_2\left(f_2\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)\right)\right\|,$$
$$d_2\left(f_2\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)\leq\left\|\left(d_1\left(f_1\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right),d_2\left(f_2\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)\right)\right\|$$

then

$$0<\rho_1\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)<\delta\quad\text{implies}\quad d_1\left(f_1\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)<\varepsilon,$$
$$0<\rho_2\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)<\delta\quad\text{implies}\quad d_2\left(f_2\left(x_{1,\alpha},x_{2,\alpha}\right),L_{i,\alpha}\right)<\varepsilon.$$

Corollary 2.5. Let $(\overline{\overline{X}},\rho)$ and $(\overline{\overline{Y}},d)$ be fuzzy metric spaces. Suppose that $f:\overline{\overline{E}}\subset\overline{\overline{X}}\longrightarrow\overline{\overline{Y}}$ and $\overline{p}$ is a fuzzy limit point of $\overline{\overline{E}}$. Then $f(\overline{x})$ converges totally to $\overline{L}\in\overline{\overline{Y}}$ as $\overline{x}\longrightarrow\overline{p}$ iff for $0<\alpha_o<\alpha_1\le\alpha\le 1$, for all $\varepsilon>0$, there exists $\delta>0$ such that

$$0<\left\|\left(\rho_1\left((x_{1,\alpha},x_{2,\alpha}),(p_{1,\alpha},p_{2,\alpha})\right),\rho_2\left((x_{1,\alpha},x_{2,\alpha}),(p_{1,\alpha},p_{2,\alpha})\right)\right)\right\|<\delta$$

implies

$$\left\|\left(d_1\left(f(x_{1,\alpha},x_{2,\alpha}),L_{i,\alpha}\right),d_2\left(f(x_{1,\alpha},x_{2,\alpha}),L_{i,\alpha}\right)\right)\right\|<\varepsilon.$$

Remark 2.6. We call the convergence in Theorem 2.4 by the partial fuzzy convergence and $\overline{L}$ by the partial fuzzy limit of f at $\overline{p}$, and write it as

$$f(\overline{p})=\overline{L}=P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x}) \tag{2.2}$$

because the limit of α-cut does not exist for $0<\alpha\le\alpha_o$. Also, we call the convergence in Corollary 2.4 by the total fuzzy convergence and $\overline{L}$ by the total fuzzy limit of f at $\overline{p}$ and write it as

$$f(\overline{p})=\overline{L}=T-\lim_{\overline{x}\to\overline{p}}f(\overline{x}) \tag{2.3}$$

because the limits of α-cuts exist for all $\alpha\in(0,1]$. Moreover, if the fuzzy convergence does not exist for all $\alpha\in(0,1]$, we denote it by $P_0-\lim_{\overline{x}\to\overline{p}}f(\overline{x})$.

Examples 2.7.

1. If $\overline{f}(\overline{x})=\overline{x}+\overline{b},\overline{x}\in\overline{\overline{\mathbb{R}}}$, then $P_1-\lim_{\overline{x}\to\overline{p}}\overline{f}(\overline{x})=f(\overline{p}),P_\alpha-\lim_{\overline{x}\to\overline{p}}\overline{f}(\overline{x})\neq f(\overline{p})$ for $\alpha\in(0,1)$ as $\overline{b}\in\overline{\overline{\mathbb{R}}}\backslash\overline{\overline{\mathbb{Z}}}$ and $T-\lim_{\overline{x}\to\overline{p}}\overline{f}(\overline{x})=f(\overline{p})$ as $\overline{b}\in\overline{\overline{\mathbb{Z}}}$ because there exists $\delta>0$ such that

$$0<\left\|\left(|x_{1,\alpha}-p_{2,\alpha}|,|x_{2,\alpha}-p_{1,\alpha}|\right)\right\|<\delta$$

Implies

$$\begin{gathered}\left\|\left(\left|f_1(x_{1,\alpha},x_{2,\alpha})-f_2(p_{1,\alpha},p_{2,\alpha})\right|,\left|f_2(x_{1,\alpha},x_{2,\alpha})-f_1(p_{1,\alpha},p_{2,\alpha})\right|\right)\right\|=\\ \left\|\left(\left|(x_{1,\alpha}+b_{1,\alpha})-(p_{2,\alpha}+b_{2,\alpha})\right|,\left|(x_{2,\alpha}-p_{2,\alpha})-(p_{1,\alpha}+b_{1,\alpha})\right|\right)\right\|\le\\ \left\|\left(|x_{1,\alpha}-p_{2,\alpha}|,|x_{2,\alpha}-p_{1,\alpha}|\right)\right\|+\left\|\left(|b_{1,\alpha}-b_{2,\alpha}|,|b_{2,\alpha}-b_{1,\alpha}|\right)\right\|<\end{gathered}$$

$$\begin{cases}\delta, & \text{if } \overline{b}\in\overline{\overline{\mathbb{Z}}} \text{ for } \alpha\in(0,1] \text{or } \overline{b}\in\overline{\overline{\mathbb{R}}}\setminus\overline{\overline{\mathbb{Z}}} \text{ for } \alpha=1,\\ \delta+\left\|\left(|b_{1,\alpha}-b_{2,\alpha}|,|b_{2,\alpha}-b_{1,\alpha}|\right)\right\|, & \text{if } \overline{b}\in\overline{\overline{\mathbb{R}}}\setminus\overline{\overline{\mathbb{Z}}} \text{ for } \alpha\in(0,1).\end{cases}$$

2. If $\overline{f}(\overline{x})=\overline{x}^2+\overline{x}-\overline{3},\overline{x}\in\overline{\overline{\mathbb{R}}}$, then $T-\lim_{\overline{x}\to\overline{1}}\overline{f}(\overline{x})=-\overline{1}$ because for all $\alpha\in(0,1]$, for all $\varepsilon>0$, there exists $0<\delta\le 1$ such that

$$0<\left\|\left(|x_{1,\alpha}-1|,|x_{2,\alpha}-1|\right)\right\|<\delta$$

implies

$$\left\|\left(\left|f_1(x_{1,\alpha},x_{2,\alpha})-f_2(1,1)\right|,\left|f_2(x_{1,\alpha},x_{2,\alpha})-f_1(1,1)\right|\right)\right\|= \left\|\left(\left|y_{1,\alpha}+x_{1,\alpha}-2\right|,\left|y_{2,\alpha}+x_{2,\alpha}-2\right|\right)\right\|<\sqrt{32}\delta=\varepsilon$$

by the Resolution Principle, where

$$y_{1,\alpha}=\min\{x_{1,\alpha}^2,x_{1,\alpha}x_{2,\alpha},x_{2,\alpha}^2\},\ y_{2,\alpha}=\max\{x_{1,\alpha}^2,x_{1,\alpha}x_{2,\alpha},x_{2,\alpha}^2\}$$

and

$$\left|y_{1,\alpha}+x_{1,\alpha}-2\right|\le\left|x_{1,\alpha}-1\right|\left|x_{1,\alpha}+2\right|<\left(\left|x_{1,\alpha}\right|+2\right)\delta<4\delta,\text{ if } y_{1,\alpha}=x_{1,\alpha}^2,$$

$$\left|y_{1,\alpha}+x_{1,\alpha}-2\right|\le\left|x_{1,\alpha}-1\right|\left|x_{2,\alpha}+1\right|+\left|x_{2,\alpha}-1\right|<\left(\left|x_{2,\alpha}\right|+1\right)\delta+\delta<4\delta,\ \text{if } y_{1,\alpha}=x_{1,\alpha}x_{2,\alpha},$$

$$\left|y_{1,\alpha}+x_{1,\alpha}-2\right|\le\left|x_{2,\alpha}^2-1\right|+\left|x_{1,\alpha}-1\right|<\left(\left|x_{2,\alpha}\right|+1\right)\delta+\delta<4\delta,\quad\text{if } y_{1,\alpha}=x_{2,\alpha}^2,$$

$$\left|y_{2,\alpha}+x_{2,\alpha}-2\right|\le\left|x_{1,\alpha}^2-1\right|+\left|x_{2,\alpha}-1\right|<\left(\left|x_{1,\alpha}\right|+1\right)\delta+\delta<4\delta,\quad\text{if } y_{2,\alpha}=x_{1,\alpha}^2,$$

$$\left|y_{2,\alpha}+x_{2,\alpha}-2\right|\le\left|x_{2,\alpha}-1\right|\left|x_{1,\alpha}+1\right|+\left|x_{1,\alpha}-1\right|<\left(\left|x_{1,\alpha}\right|+1\right)\delta+\delta<4\delta,\ \text{if } y_{2,\alpha}=x_{1,\alpha}x_{2,\alpha},$$

$$\left|y_{2,\alpha}+x_{2,\alpha}-2\right|\le\left|x_{2,\alpha}-1\right|\left|x_{2,\alpha}+2\right|<\left(\left|x_{2,\alpha}\right|+2\right)\delta<4\delta,\text{ if } y_{2,\alpha}=x_{2,\alpha}^2.$$

Set $\delta=\min\left\{1,\frac{\varepsilon}{\sqrt{32}}\right\}$, we complete the proof.

The next step is to consider the fundamental properties of fuzzy limits of fuzzy functions by generalizing the basic properties of the classical ones.

Theorem 2.8. The partial fuzzy limit of a fuzzy function is unique if it exists.

Proof. Suppose $f:\overline{\overline{E}}\subset\overline{\overline{X}}\longrightarrow\overline{\overline{Y}}$ and $\overline{p}\in\overline{\overline{X}}$ is a fuzzy limit point of $\overline{\overline{E}}$. Assume that $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=\overline{L}, P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=\overline{M}$. So, by the Resolution principle, for all $0<\alpha_o<\alpha_1\le\alpha\le1$, for all $\varepsilon>0$, there exist $\delta_1,\delta_2>0$ such that

$$0<\left\|\left(\rho_1\left((x_{1,\alpha},x_{2,\alpha}),(p_{1,\alpha},p_{2,\alpha})\right),\rho_2\left((x_{1,\alpha},x_{2,\alpha}),(p_{1,\alpha},p_{2,\alpha})\right)\right)\right\|<\delta_1$$

implies

$$\left\|\left(d_1\left(f_1(x_{1,\alpha},x_{2,\alpha}),L_{i,\alpha}\right),d_2\left(f_2(x_{1,\alpha},x_{2,\alpha}),L_{i,\alpha}\right)\right)\right\|<\frac{\varepsilon}{2}$$

and

$$0<\left\|\left(\rho_1\left((x_{1,\alpha},x_{2,\alpha}),(p_{1,\alpha},p_{2,\alpha})\right),\rho_2\left((x_{1,\alpha},x_{2,\alpha}),(p_{1,\alpha},p_{2,\alpha})\right)\right)\right\|<\delta_2$$

implies

$$\left\|\left(d_1\left(f_1(x_{1,\alpha},x_{2,\alpha}),M_{i,\alpha}\right),d_2\left(f_2(x_{1,\alpha},x_{2,\alpha}),M_{i,\alpha}\right)\right)\right\|<\frac{\varepsilon}{2}.$$

Let $\delta=\min\{\delta_1,\delta_2\}$. Then,

$$0<\left\|\left(\rho_1\left((x_{1,\alpha},x_{2,\alpha}),(p_{1,\alpha},p_{2,\alpha})\right),\rho_2\left((x_{1,\alpha},x_{2,\alpha}),(p_{1,\alpha},p_{2,\alpha})\right)\right)\right\|<\delta$$

implies

$$\left\|\left(d_1(L_{i,\alpha}, M_{i,\alpha}), d_2(L_{i,\alpha}, M_{i,\alpha})\right)\right\| \leq$$
$$\left\|\left(d_1\left(L_{i,\alpha}, f_1(x_{1,\alpha}, x_{2,\alpha})\right), d_2\left(L_{i,\alpha}, f_2(x_{1,\alpha}, x_{2,\alpha})\right)\right)\right\|$$
$$+\left\|\left(d_1(f_1(x_{1,\alpha}, x_{2,\alpha}), M_{i,\alpha}), d_2(f_2(x_{1,\alpha}, x_{2,\alpha}), M_{i,\alpha})\right)\right\| < \varepsilon,$$

where

$$d_1(L_{i,\alpha}, M_{i,\alpha}) = \min\{d(L_{i,\alpha}, M_{i,\alpha}): i = 1,2\},$$
$$d_2(L_{i,\alpha}, M_{i,\alpha}) = \max\{d(L_{i,\alpha}, M_{i,\alpha}): i = 1,2\}.$$

Corollary 2.9. The total fuzzy limit of a fuzzy function is unique if it exists.

Theorem 2.10. Let $f: \overline{\overline{E}} \subset \overline{\overline{X}} \longrightarrow \overline{\overline{Y}}, \overline{p}$ be a fuzzy limit point of $\overline{\overline{E}}$ and $\overline{L} \in \overline{\overline{Y}}$. Then $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = \overline{L}$ iff $P_{\alpha_1} - \lim_{n\to\infty} f(\overline{p}_n) = \overline{L}$ for every fuzzy sequence $\overline{p}_n$ in $\overline{\overline{E}}$ such that $\overline{p}_n \neq \overline{p}$, $\lim_{n\to\infty} \overline{p}_n = \overline{p}$.

Proof. Suppose that $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = \overline{L}$ holds. By the Resolution principle, for all $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \left\|\left(\rho_1\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right), \rho_2\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right)\right)\right\| < \delta$$

implies

$$\left\|\left(d_1(f_1(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}), d_2(f_2(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha})\right)\right\| < \varepsilon.$$

Since $\overline{p}_n \longrightarrow \overline{p}$, then for $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, there exits $N \in \mathbb{N}$ such that for $n > N$,

$$0 < \left\|\left(\rho_1\left((p_{n,1,\alpha}, p_{n,2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right), \rho_2\left((p_{n,1,\alpha}, p_{n,2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right)\right)\right\| < \delta$$

implies

$$\left\|\left(d_1(f_1(p_{n,1,\alpha}, p_{n,2,\alpha}), L_{i,\alpha}), d_2(f_2(p_{n,1,\alpha}, p_{n,2,\alpha}), L_{i,\alpha})\right)\right\| < \varepsilon.$$

Conversely, assume that $P_{\alpha_1} - \lim_{n\to\infty} f(\overline{p}_n) = \overline{L}$, but $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) \neq \overline{L}$. So, for $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, there exists $\varepsilon_o > 0$, for every $\delta > 0$, such that

$$0 < \left\|\left(\rho_1\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right), \rho_2\left((x_{1,\alpha}, x_{2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right)\right)\right\| < \delta$$

but

$$\left\|\left(d_1(f_1(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha}), d_2(f_2(x_{1,\alpha}, x_{2,\alpha}), L_{i,\alpha})\right)\right\| > \varepsilon_o$$

Taking $\delta = \frac{1}{n}, n \in \mathbb{N}$, there is a $\overline{p}_n$ in $\overline{\overline{E}}$ such that

$$0 < \left\|\left(\rho_1\left((p_{n,1,\alpha}, p_{n,2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right), \rho_2\left((p_{n,1,\alpha}, p_{n,2,\alpha}), (p_{1,\alpha}, p_{2,\alpha})\right)\right)\right\| < \frac{1}{n}$$

implies

$$\left\|\left(d_1\left(f_1\left(p_{n,1,\alpha},p_{n,2,\alpha}\right),L_{i,\alpha}\right),d_2\left(f_2\left(p_{n,1,\alpha},p_{n,2,\alpha}\right),L_{i,\alpha}\right)\right)\right\|>\varepsilon_o$$

which contradicts the assumption $P_{\alpha_1}-\lim_{n\to\infty}f(\overline{p}_n)=\overline{L}$.

Corollary 2.11. Let $f:\overline{\overline{E}}\subset\overline{\overline{X}}\longrightarrow\overline{\overline{Y}},\overline{p}$ be a fuzzy limit point of $\overline{\overline{E}}$ and $\overline{L}\in\overline{\overline{Y}}$. Then $T-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=\overline{L}$ iff $T-\lim_{n\to\infty}f(\overline{p}_n)=\overline{L}$ for every fuzzy sequence $\overline{p}_n$ in $\overline{\overline{E}}$ such that $\overline{p}_n\neq\overline{p}$, $\lim_{n\to\infty}\overline{p}_n=\overline{p}$.

Theorem 2.12. If f and g are fuzzy functions such that $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}g(\overline{x})=\overline{L}$ and $P_{\alpha_2}-\lim_{\overline{u}\to\overline{L}}f(\overline{u})=f(\overline{L})$, then $P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}f(g(\overline{x}))=f\left(P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}g(\overline{x})\right)=f(\overline{L})$, $\alpha_M=\max\{\alpha_1,\alpha_2\}$.

Proof. Since $f(\overline{u})\longrightarrow f(\overline{L})$ as $\overline{u}\to\overline{L}$ partially for $\alpha\geq\alpha_2$, then by the Resolution Principle, for all $\varepsilon>0$, there exits $\delta>0$ such that

$$0<\left\|\left(\rho_1\left(\left(u_{1,\alpha},u_{2,\alpha}\right),\left(L_{1,\alpha},L_{2,\alpha}\right)\right),\rho_2\left(\left(u_{1,\alpha},u_{2,\alpha}\right),\left(L_{1,\alpha},L_{2,\alpha}\right)\right)\right)\right\|<\delta$$

implies

$$\left\|\left(d_1\left(f_1\left(u_{1,\alpha},u_{2,\alpha}\right),f_i\left(L_{1,\alpha},L_{2,\alpha}\right)\right),d_2\left(f_2\left(u_{1,\alpha},u_{2,\alpha}\right),f_i\left(L_{1,\alpha},L_{2,\alpha}\right)\right)\right)\right\|<\varepsilon.$$

Since $g(\overline{x})\longrightarrow\overline{L}$ as $\overline{x}\longrightarrow\overline{p}$ partially for $\alpha\geq\alpha_1$, then by the Resolution Principle, there exists $\delta'>0$ such that

$$0<\left\|\left(\sigma_1\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right),\sigma_2\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)\right)\right\|<\delta'$$

implies

$$\left\|\left(\rho_1\left(g_1\left(x_{1,\alpha},x_{2,\alpha}\right),g_i\left(p_{1,\alpha},p_{2,\alpha}\right)\right),\rho_2\left(g_2\left(x_{1,\alpha},x_{2,\alpha}\right),g_i\left(p_{1,\alpha},p_{2,\alpha}\right)\right)\right)\right\|<\delta.$$

Letting $\left(u_{1,\alpha},u_{2,\alpha}\right)=\left(g_1\left(x_{1,\alpha},x_{2,\alpha}\right),g_2\left(x_{1,\alpha},x_{2,\alpha}\right)\right)$ for $0<\alpha_o<\alpha_M\leq\alpha\leq1$,we obtain

$$0<\left\|\left(\sigma_1\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right),\sigma_2\left(\left(x_{1,\alpha},x_{2,\alpha}\right),\left(p_{1,\alpha},p_{2,\alpha}\right)\right)\right)\right\|<\delta'$$

implies

$$\left\|\begin{pmatrix}d_1\left(f_1\left(g_1\left(x_{1,\alpha},x_{2,\alpha}\right),g_2\left(x_{1,\alpha},x_{2,\alpha}\right)\right),f_i\left(L_{1,\alpha},L_{2,\alpha}\right)\right),\\ d_2\left(f_2\left(g_1\left(x_{1,\alpha},x_{2,\alpha}\right),g_2\left(x_{1,\alpha},x_{2,\alpha}\right)\right),f_i\left(L_{1,\alpha},L_{2,\alpha}\right)\right)\end{pmatrix}\right\|<\varepsilon.$$

Corollary 2.13. Let f and g be fuzzy functions.

1. If $T-\lim_{\overline{x}\to\overline{p}} g(\overline{x})=\overline{L}$ and $P_{\alpha_1}-\lim_{\overline{u}\to\overline{L}} f(\overline{u})=f(\overline{L})$, then
$$P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} f(g(\overline{x}))= \quad f\left(P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} g(\overline{x})\right)=f(\overline{L}).$$
2. If $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} g(\overline{x})=\overline{L}$ and $T-\lim_{\overline{u}\to\overline{L}} f(\overline{u})=f(\overline{L})$, then
$$P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} f(g(\overline{x}))=f\left(P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} g(\overline{x})\right)=f(\overline{L}).$$
3. If $T-\lim_{\overline{x}\to\overline{p}} g(\overline{x})=\overline{L}$ and $T-\lim_{\overline{u}\to\overline{L}} f(\overline{u})=f(\overline{L})$, then
$$T-\lim_{\overline{x}\to\overline{p}} f(g(\overline{x}))=f\left(T-\lim_{\overline{x}\to\overline{p}} g(\overline{x})\right)=f(\overline{L}).$$

Theorem 2.14. If $\overline{\overline{E}}\subset\overline{\overline{\mathbb{R}}}$ is a fuzzy metric space, $\overline{p}$ is a fuzzy limit point of $\overline{\overline{E}}$ and f,g are fuzzy functions on $\overline{\overline{E}}$, then

1. $\left(P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} f(\overline{x})\right)+\left(P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}} g(\overline{x})\right)=P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}(f(\overline{x})+g(\overline{x})), \alpha_M=\max\{\alpha_1,\alpha_2\}$.
2. $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}(\overline{A}f)(\overline{x})=\overline{A}\left(P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} f(\overline{x})\right), \overline{A}\in\overline{\overline{R}}$.
3. $\left(P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} f(\overline{x})\right)\left(P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}} g(\overline{x})\right)=P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}(fg)(\overline{x}), \alpha_M=\max\{\alpha_1,\alpha_2\}$.
4. $\frac{P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} f(\overline{x})}{P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}} g(\overline{x})}=P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}\left(\frac{f(\overline{x})}{g(\overline{x})}\right), \alpha_M=\max\{\alpha_1,\alpha_2\}$.

Proof. To prove (1), by the Resolution Principle, we have

$$\begin{aligned}
&P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}} f(\overline{x})+P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}} g(\overline{x})\\
&=\bigcup_{\alpha\in[\alpha_1,1]}\left(\alpha\left[\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}} f_1(x_{1,\alpha},x_{2,\alpha}), \lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}} f_2(x_{1,\alpha},x_{2,\alpha})\right]\right)\\
&+\bigcup_{\alpha\in[\alpha_2,1]}\left(\alpha\left[\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}} g_1(x_{1,\alpha},x_{2,\alpha}), \lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}} g_2(x_{1,\alpha},x_{2,\alpha})\right]\right)\\
&=\bigcup_{\alpha\in[\alpha_M,1]}\left(\alpha\left[\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}\left(f_1(x_{1,\alpha},x_{2,\alpha})+g_1(x_{1,\alpha},x_{2,\alpha})\right), \lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}\left(f_2(x_{1,\alpha},x_{2,\alpha})\right.\right.\right.\\
&\left.\left.\left.+g_2(x_{1,\alpha},x_{1,\alpha})\right)\right]\right)=P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}\left(\overline{f}(\overline{x})+\overline{g}(\overline{x})\right).
\end{aligned}$$

To prove (2), by the Resolution Principle, we have

$$P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}(\overline{A}f)(\overline{x})=\bigcup_{\alpha\in[\alpha_1,1]}\left(\alpha\left[\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}F_1(x_{1,\alpha},x_{2,\alpha}),\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}F_2(x_{1,\alpha},x_{2,\alpha})\right]\right)$$
$$=\bigcup_{\alpha\in(0,1]}(\alpha[A_{1,\alpha},A_{2,\alpha}])\bigcup_{\alpha\in[\alpha_1,1]}\left(\alpha\left[\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}f_1(x_{1,\alpha},x_{2,\alpha}),\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}f_2(x_{1,\alpha},x_{2,\alpha})\right]\right)$$
$$=\overline{A}\left(P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})\right),$$

where

$$F_1(x_{1,\alpha},x_{2,\alpha})$$
$$=\min\{A_{1,\alpha}f_1(x_{1,\alpha},x_{2,\alpha}),A_{1,\alpha}f_2(x_{1,\alpha},x_{2,\alpha}),A_{2,\alpha}f_1(x_{1,\alpha},x_{2,\alpha}),A_{2,\alpha}f_2(x_{1,\alpha},x_{2,\alpha})\},$$
$$F_2(x_{1,\alpha},x_{2,\alpha})$$
$$=\max\{A_{1,\alpha}f_1(x_{1,\alpha},x_{2,\alpha}),A_{1,\alpha}f_2(x_{1,\alpha},x_{2,\alpha}),A_{2,\alpha}f_1(x_{1,\alpha},x_{2,\alpha}),A_{2,\alpha}f_2(x_{1,\alpha},x_{2,\alpha})\}.$$

To prove (3), let $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=\overline{L}$ and $P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}}g(\overline{x})=\overline{M}$, then $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}[f(\overline{x})-\overline{L}]=\overline{0}$ and $P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}}[g(\overline{x})-\overline{M}]=\overline{0}$. By the Resolution Principle, for $\alpha_M\le\alpha\le 1$, for all $\varepsilon>0$, there exists $\delta>0$, such that

$$0<\left\|\left(|x_{1,\alpha}-p_{2,\alpha}|_1,|x_{2,\alpha}-p_{1,\alpha}|\right)\right\|<\delta$$

Implies

$$\|(|f_1(x_{1,\alpha},x_{2,\alpha})-L_{2,\alpha}|,|f_2(x_{1,\alpha},x_{2,\alpha})-L_{1,\alpha}|)\|<\varepsilon$$

and

$$0<\left\|\left(|x_{1,\alpha}-p_{2,\alpha}|_1,|x_{2,\alpha}-p_{1,\alpha}|\right)\right\|<\delta$$

Implies

$$\|(|g_1(x_{1,\alpha},x_{2,\alpha})-M_{2,\alpha}|,|g_2(x_{1,\alpha},x_{2,\alpha})-M_{1,\alpha}|)\|<\varepsilon.$$

So,

$$\|(|(FG)_1|,|(FG)_2|)\|\le\|(|F_1|,|F_2|)\|\|(|G_1|,|G_2|)\|<\varepsilon,$$

where

$$F_1=f_1(x_{1,\alpha},x_{2,\alpha})-L_{2,\alpha},F_2=f_2(x_{1,\alpha},x_{2,\alpha})-L_{1,\alpha},$$
$$G_1=g_1(x_{1,\alpha},x_{2,\alpha})-M_{2,\alpha},G_2=g_2(x_{1,\alpha},x_{2,\alpha})-M_{1,\alpha},$$
$$(FG)_1=\min\{F_1G_1,F_1G_2,F_2G_1,F_2G_2\},$$
$$(FG)_2=\max\{F_1G_1,F_1G_2,F_2G_1,F_2G_2\}.$$

That is,

$$\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}(FG)_1=0,\lim_{\substack{x_{1,\alpha}\to p_{2,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}(FG)_2=0.$$

From properties (1) and (2), if

$$f_1(x_{1,\alpha},x_{2,\alpha})g_1(x_{1,\alpha},x_{2,\alpha}) = \min\{f_i(x_{1,\alpha},x_{2,\alpha})g_i(x_1,x_{2,\alpha}): i = 1,2\}$$

or

$$f_1(x_{1,\alpha},x_{2,\alpha})g_1(x_{1,\alpha},x_{2,\alpha}) = \max\{f_i(x_{1,\alpha},x_{2,\alpha})g_i(x_1,x_{2,\alpha}): i = 1,2\},$$

Then

$$\begin{aligned}\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}} & f_1(x_{1,\alpha},x_{2,\alpha})g_1(x_{1,\alpha},x_{2,\alpha}) \\ &= \lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}\big([f_1(x_{1,\alpha},x_{2,\alpha}) - L_{2,\alpha}][g_1(x_{1,\alpha},x_{2,\alpha}) - M_{2,\alpha}] \\ &\quad + L_{2,\alpha}g_1(x_{1,\alpha},x_{2,\alpha}) + M_{2,\alpha}f_1(x_{1,\alpha},x_{2,\alpha}) - L_{2,\alpha}M_{2,\alpha}\big) \\ &= 0 + L_{2,\alpha}M_{2,\alpha} + L_{2,\alpha}M_{2,\alpha} - L_{2,\alpha}M_{2,\alpha} = L_{2,\alpha}M_{2,\alpha}.\end{aligned}$$

If

$$f_1(x_{1,\alpha},x_{2,\alpha})g_2(x_{1,\alpha},x_{2,\alpha}) = \min\{f_i(x_{1,\alpha},x_{2,\alpha})g_i(x_1,x_{2,\alpha}): i = 1,2\}$$

or

$$f_1(x_{1,\alpha},x_{2,\alpha})g_2(x_{1,\alpha},x_{2,\alpha}) = \max\{f_i(x_{1,\alpha},x_{2,\alpha})g_i(x_{1,\alpha},x_{2,\alpha}): i = 1,2\},$$

then

$$\begin{aligned}\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}} & f_1(x_{1,\alpha},x_{2,\alpha})g_2(x_{1,\alpha},x_{2,\alpha}) \\ &= \lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}\big([f_1(x_{1,\alpha},x_{2,\alpha}) - L_{2,\alpha}][g_2(x_{1,\alpha},x_{2,\alpha}) - M_{1,\alpha}] \\ &\quad + L_{2,\alpha}g_2(x_{1,\alpha},x_{2,\alpha}) + M_{1,\alpha}f_1(x_{1,\alpha},x_{2,\alpha}) - L_{2,\alpha}M_{1,\alpha}\big) \\ &= 0 + L_{2,\alpha}M_{1,\alpha} + L_{2,\alpha}M_{1,\alpha} - L_{2,\alpha}M_{1,\alpha} = L_{2,\alpha}M_{1,\alpha}.\end{aligned}$$

If

$$f_2(x_{1,\alpha},x_{2,\alpha})g_1(x_{1,\alpha},x_{2,\alpha}) = \min\{f_i(x_{1,\alpha},x_{2,\alpha})g_i(x_1,x_{2,\alpha}): i = 1,2\}$$

or

$$f_2(x_{1,\alpha},x_{2,\alpha})g_1(x_{1,\alpha},x_{2,\alpha}) = \max\{f_i(x_{1,\alpha},x_{2,\alpha})g_i(x_1,x_{2,\alpha}): i = 1,2\},$$

then

$$\begin{aligned}\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}} & f_2(x_{1,\alpha},x_{2,\alpha})g_1(x_{1,\alpha},x_{2,\alpha}) \\ &= \lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}}\big([f_2(x_{1,\alpha},x_{2,\alpha}) - L_{1,\alpha}][g_1(x_{1,\alpha},x_{2,\alpha}) - M_{2,\alpha}] \\ &\quad + L_{1,\alpha}g_1(x_{1,\alpha},x_{2,\alpha}) + M_{2,\alpha}f_2(x_{1,\alpha},x_{2,\alpha}) - L_{1,\alpha}M_{2,\alpha}\big) \\ &= 0 + L_{1,\alpha}M_{2,\alpha} + L_{1,\alpha}M_{2,\alpha} - L_{1,\alpha}M_{2,\alpha} = L_{1,\alpha}M_{2,\alpha}.\end{aligned}$$

If

$$f_2(x_{1,\alpha},x_{2,\alpha})g_2(x_{1,\alpha},x_{2,\alpha}) = \min\{f_i(x_{1,\alpha},x_{2,\alpha})g_i(x_{1,\alpha},x_{2,\alpha}): i = 1,2\}$$

or

$$f_2(x_{1,\alpha}, x_{2,\alpha})g_2(x_{1,\alpha}, x_{2,\alpha}) = \max\{f_i(x_{1,\alpha}, x_{2,\alpha})g_i(x_1, x_{2,\alpha}) : i = 1,2\},$$

then

$$\begin{aligned} &\lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}} f_2(x_{1,\alpha}, x_{2,\alpha})g_2(x_{1,\alpha}, x_{2,\alpha}) \\ &\qquad = \lim_{\substack{x_{1,\alpha}\to p_{1,\alpha}\\ x_{2,\alpha}\to p_{2,\alpha}}} \left([f_2(x_{1,\alpha}, x_{2,\alpha}) - L_{1,\alpha}][g_2(x_{1,\alpha}, x_{2,\alpha}) - M_{1,\alpha}]\right. \\ &\qquad \left. + L_{2,\alpha}g_2(x_{1,\alpha}, x_{2,\alpha}) + M_{1,\alpha}f_2(x_{1,\alpha}, x_{2,\alpha}) - L_{1,\alpha}M_{1,\alpha}\right) \\ &\qquad = 0 + L_{1,\alpha}M_{1,\alpha} + L_{1,\alpha}M_{1,\alpha} - L_{1,\alpha}M_{1,\alpha} = L_{1,\alpha}M_{1,\alpha}. \end{aligned}$$

In order to prove (4), since $P_{\alpha_2} - \lim_{\overline{x}\to\overline{p}} g(\overline{x}) = \overline{M}$, then by the Resolution Principle, for $\alpha \geq \alpha_2$, for all $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$0 < \|(|x_{1,\alpha} - p_{2,\alpha}|, |x_{2,\alpha} - p_{1,\alpha}|)\| < \delta_1$$

implies

$$\|(|g_1(x_{1,\alpha}, x_{2,\alpha}) - M_{2,\alpha}|, |g_2(x_{1,\alpha}, x_{2,\alpha}) - M_{1,\alpha}|)\| < \varepsilon.$$

So,

$$0 < \|(|x_{1,\alpha} - p_{2,\alpha}|, |x_{2,\alpha} - p_{1,\alpha}|)\| < \delta_1$$

implies

$$\|(|g_1(x_{1,\alpha}, x_{2,\alpha}) - M_{2,\alpha}|, |g_2(x_{1,\alpha}, x_{2,\alpha}) - M_{1,\alpha}|)\| < \frac{\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\|}{2}$$

which involves

$$\begin{aligned} \|(|M_{1,\alpha}|, |M_{2,\alpha}|)\| &\leq \|(|g_1(x_{1,\alpha}, x_{2,\alpha})|, |g_2(x_{1,\alpha}, x_{2,\alpha})|)\| \\ &\quad + \|(|g_1(x_{1,\alpha}, x_{2,\alpha}) - M_{2,\alpha}|, |g_2(x_{1,\alpha}, x_{2,\alpha}) - M_{1,\alpha}|)\| \\ &< \|(|g_1(x_{1,\alpha}, x_{2,\alpha}) - M_{2,\alpha}|, |g_2(x_{1,\alpha}, x_{2,\alpha}) - M_{1,\alpha}|)\| \\ &\quad + \frac{\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\|}{2}. \end{aligned}$$

Rearranging above, we obtain

$$\frac{1}{\|(|g_1(x_{1,\alpha}, x_{2,\alpha})|, |g_2(x_{1,\alpha}, x_{2,\alpha})|)\|} < \frac{2}{\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\|}.$$

Moreover, there exists $\delta_2 > 0$ such that

$$0 < \|(|x_{1,\alpha} - p_{2,\alpha}|, |x_{2,\alpha} - p_{1,\alpha}|)\| < \delta_2$$

implies

$$\|(|g_1(x_{1,\alpha}, x_{2,\alpha}) - M_{2,\alpha}|, |g_2(x_{1,\alpha}, x_{2,\alpha}) - M_{1,\alpha}|)\| < \frac{\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\|^2 \varepsilon}{2}.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then

$$0 < \|(|x_{1,\alpha} - p_{2,\alpha}|, |x_{2,\alpha} - p_{1,\alpha}|)\| < \delta$$

implies

$$\left|\frac{1}{\|(|g_1(x_{1,\alpha}, x_{2,\alpha})|, |g_2(x_{1,\alpha}, x_{2,\alpha})|)\|} - \frac{1}{\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\|}\right|$$
$$= \frac{|\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\| - \|(|g_1(x_{1,\alpha}, x_{2,\alpha})|, |g_2(x_{1,\alpha}, x_{2,\alpha})|)\||}{\|(|g_1(x_{1,\alpha}, x_{2,\alpha})|, |g_2(x_{1,\alpha}, x_{2,\alpha})|)\|\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\|}$$
$$\leq \frac{\|(|g_1(x_{1,\alpha}, x_{2,\alpha}) - M_{2,\alpha}|, |g_2(x_{1,\alpha}, x_{2,\alpha}) - M_{1,\alpha}|)\|}{\|(|g_1(x_{1,\alpha}, x_{2,\alpha})|, |g_2(x_{1,\alpha}, x_{2,\alpha})|)\|\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\|}$$
$$< \frac{2}{\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\|^2} \frac{\|(|M_{1,\alpha}|, |M_{2,\alpha}|)\|^2 \varepsilon}{2} = \varepsilon.$$

Using (3), we complete the proof.

Corollary 2.15. If $\overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}}$ is a fuzzy metric space, $\overline{p}$ is a fuzzy limit point of $\overline{\overline{E}}$ and f, g are fuzzy functions on $\overline{\overline{E}}$, then

1. $T - \lim_{\overline{x} \to \overline{p}} f(\overline{x}) + P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} g(\overline{x}) = P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} \big(f(\overline{x}) + g(\overline{x})\big) = P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} f(\overline{x}) + T - \lim_{\overline{x} \to \overline{p}} g(\overline{x})$

2. $T - \lim_{\overline{x} \to \overline{p}} f(\overline{x}) + T - \lim_{\overline{x} \to \overline{p}} g(\overline{x}) = T - \lim_{\overline{x} \to \overline{p}} \big(f(\overline{x}) + g(\overline{x})\big)$

3. $T - \lim_{\overline{x} \to \overline{p}} (\overline{A} f)(\overline{x}) = \overline{A}\left(T - \lim_{\overline{x} \to \overline{p}} f(\overline{x})\right), \overline{A} \in \overline{\overline{R}}$

4. $\left(T - \lim_{\overline{x} \to \overline{p}} f(\overline{x}))\right)\left(P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} g(\overline{x})\right) = P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} (fg)(\overline{x}) = \left(P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} f(\overline{x})\right)\left(T - \lim_{\overline{x} \to \overline{p}} g(\overline{x})\right)$

5. $\left(T - \lim_{\overline{x} \to \overline{p}} f(\overline{x})\right)\left(T - \lim_{\overline{x} \to \overline{p}} g(\overline{x})\right) = T - \lim_{\overline{x} \to \overline{p}} (fg)(\overline{x})$

6. $\frac{T - \lim_{\overline{x} \to \overline{p}} f(\overline{x})}{P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} g(\overline{x})} = P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} \left(\frac{f(\overline{x})}{g(\overline{x})}\right) = \frac{P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} f(\overline{x})}{T - \lim_{\overline{x} \to \overline{p}} g(\overline{x})}$

7. $\frac{T - \lim_{\overline{x} \to \overline{p}} f(\overline{x})}{T - \lim_{\overline{x} \to \overline{p}} g(\overline{x})} = T - \lim_{\overline{x} \to \overline{p}} \left(\frac{f(\overline{x})}{g(\overline{x})}\right).$

Theorem 2.16. Let $\overline{p} \in \overline{\overline{I}} \subset \overline{\overline{R}}$, where $\overline{\overline{I}}$ is an open fuzzy interval. If f, g are fuzzy functions defined on $\overline{\overline{I}}\backslash\overline{p}$ such that $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x})$ and $P_{\alpha_2} - \lim_{\overline{x}\to\overline{p}} g(\overline{x})$ exist and $f(\overline{x}) = g(\overline{x}), \overline{x} \in \overline{\overline{I}}\backslash\overline{p}$, then $P_{\alpha_M} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = P_{\alpha_M} - \lim_{\overline{x}\to\overline{a}} g(\overline{x}), \alpha_M = \max\{\alpha_1, \alpha_2\}$.

Corollary 2.17. Let $\overline{p} \in \overline{\overline{I}} \subset \overline{\overline{R}}$, where $\overline{\overline{I}}$ is an open fuzzy interval and f, g are fuzzy functions defined on $\overline{\overline{I}}\backslash\overline{p}$ such that and $f(\overline{x}) = g(\overline{x}), \overline{x} \in \overline{\overline{I}}\backslash\overline{p}$. We have the following

1. If $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x})$ exists, then $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = P_{\alpha_1} - \lim_{\overline{x}\to\overline{a}} g(\overline{x})$.
2. If $P_{\alpha_2} - \lim_{\overline{x}\to\overline{p}} g(\overline{x})$ exists, then $P_{\alpha_2} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = P_{\alpha_2} - \lim_{\overline{x}\to\overline{a}} g(\overline{x})$.
3. If $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x})$ and $T - \lim_{\overline{x}\to\overline{p}} g(\overline{x})$ exist, then $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = P_{\alpha_1} - \lim_{\overline{x}\to\overline{a}} g(\overline{x})$.
4. If $T - \lim_{\overline{x}\to\overline{p}} f(\overline{x})$ and $P_{\alpha_2} - \lim_{\overline{x}\to\overline{p}} g(\overline{x})$ exist, then $P_{\alpha_2} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = P_{\alpha_2} - \lim_{\overline{x}\to\overline{a}} g(\overline{x})$.
5. If $T - \lim_{\overline{x}\to\overline{p}} f(\overline{x})$ and $T - \lim_{\overline{x}\to\overline{p}} g(\overline{x})$ exist, then $T - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = T - \lim_{\overline{x}\to\overline{a}} g(\overline{x})$.

Theorem 2.18. Suppose $\overline{p} \in \overline{\overline{I}} \subset \overline{\overline{R}}$, where $\overline{\overline{I}}$ is an open fuzzy interval, and f, g are fuzzy functions defined on $\overline{\overline{I}}\backslash\overline{p}$. If $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x})$ and $P_{\alpha_2} - \lim_{\overline{x}\to\overline{p}} g(\overline{x})$ exist and $f(\overline{x}) \leq g(\overline{x}), \overline{x} \in \overline{\overline{I}}\backslash\overline{p}$, then $P_{\alpha_M} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) \leq P_{\alpha_M} - \lim_{\overline{x}\to\overline{p}} g(\overline{x}), \alpha_M = \max\{\alpha_1, \alpha_2\}$.

Proof. Let $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = \overline{L}$ and $P_{\alpha_2} - \lim_{\overline{x}\to\overline{p}} g(\overline{x}) = \overline{M}$. Assume, by the Resolution Principle, for $\alpha_M \leq \alpha \leq 1$, that $[L_{1,\alpha}, L_{2,\alpha}] > [M_{1,\alpha}, M_{2,\alpha}]$. For $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 = \frac{1}{2}[L_{1,\alpha} - M_{2,\alpha}]$, there exist $\delta_1, \delta_2 > 0$ such that

$$0 < \left\|\left(\left|x_{1,\alpha} - p_{2,\alpha}\right|, \left|x_{2,\alpha} - p_{1,\alpha}\right|\right)\right\| < \delta_1$$

Implies

$$\left\|\left(\left|f_1(x_{1,\alpha}, x_{2,\alpha}) - L_{2,\alpha}\right|, \left|f_2(x_{1,\alpha}, x_{2,\alpha}) - L_{1,\alpha}\right|\right)\right\| < \varepsilon_1$$

And

$$0 < \left\|\left(\left|x_{1,\alpha} - p_{2,\alpha}\right|, \left|x_{2,\alpha} - p_{1,\alpha}\right|\right)\right\| < \delta_2$$

Implies

$$\left\|\left(\left|g_1(x_{1,\alpha}, x_{2,\alpha}) - M_{2,\alpha}\right|, \left|g_2(x_{1,\alpha}, x_{2,\alpha}) - M_{1,\alpha}\right|\right)\right\| < \varepsilon_2.$$

Letting $\delta = \min\{\delta_1, \delta_2\}$, we get

$$0 < \left\|\left(\left|x_{1,\alpha} - p_{2,\alpha}\right|, \left|x_{2,\alpha} - p_{1,\alpha}\right|\right)\right\| < \delta$$

implies

$$\begin{aligned}\left(f_1(x_{1,\alpha},x_{2,\alpha})-g_2(x_{1,\alpha},x_{2,\alpha}),f_2(x_{1,\alpha},x_{2,\alpha})-g_1(x_{1,\alpha},x_{2,\alpha})\right)\\ =\left(f_1(x_{1,\alpha},x_{2,\alpha})-L_{2,\alpha},f_2(x_{1,\alpha},x_{2,\alpha})-L_{1,\alpha}\right)\\ +\left(L_{2,\alpha}-M_{1,\alpha},L_{1,\alpha}-M_{2,\alpha}\right)\\ +\left(M_{1,\alpha}-g_2(x_{1,\alpha},x_{2,\alpha}),M_{2,\alpha}-g_1(x_{1,\alpha},x_{2,\alpha})\right)\\ >\left(L_{2,\alpha}-M_{1,\alpha}-\varepsilon_1-\varepsilon_2,L_{1,\alpha}-M_{2,\alpha}-\varepsilon_1-\varepsilon_2\right)>(0,0)\end{aligned}$$

which contradicts the assumption $\overline{f}(\overline{x})\leq\overline{g}(\overline{x}),\overline{x}\in\overline{\overline{I}}\backslash\overline{p}$.

Corollary 2.19. Suppose $\overline{p}\in\overline{\overline{I}}\subset\overline{\overline{R}}$, where $\overline{\overline{I}}$ is an open fuzzy interval, and f,g are fuzzy functions defined on $\overline{\overline{I}}\backslash\overline{p}$ such that $f(\overline{x})\leq g(\overline{x})$ for all $\overline{x}\in\overline{\overline{I}}\backslash\overline{p}$. We have the following

1. If $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})$ exists, then $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})\leq P_{\alpha_1}-\lim_{\overline{x}\to\overline{a}}g(\overline{x})$.
2. If $P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}}g(\overline{x})$ exists, then $P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})\leq P_{\alpha_2}-\lim_{\overline{x}\to\overline{a}}g(\overline{x})$.
3. If $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})$ and $T-\lim_{\overline{x}\to\overline{p}}g(\overline{x})$ exist, then $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})\leq P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}g(\overline{x})$.
4. If $T-\lim_{\overline{x}\to\overline{p}}f(\overline{x})$ and $P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}}g(\overline{x})$ exist, then $P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})\leq P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}}g(\overline{x})$.
5. If $T-\lim_{\overline{x}\to\overline{p}}f(\overline{x})$ and $T-\lim_{\overline{x}\to\overline{p}}g(\overline{x})$ exist, then $T-\lim_{\overline{x}\to\overline{p}}f(\overline{x})\leq T-\lim_{\overline{x}\to\overline{p}}g(\overline{x})$.

Theorem 2.20. Suppose $\overline{p}\in\overline{\overline{I}}\subset\overline{\overline{R}}$, where $\overline{\overline{I}}$ is an open fuzzy interval, and f,g,h are fuzzy functions such that $f(\overline{x})\leq h(\overline{x})\leq g(\overline{x})$ for all $\overline{x}\in\overline{\overline{I}}\backslash\overline{p}$ and $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})$, $P_{\alpha_2}-\lim_{\overline{x}\to\overline{p}}g(\overline{x})$ exist. Then

$$P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}h(\overline{x})=P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}g(\overline{x}),\alpha_M=\max\{\alpha_1,\alpha_2\}.$$

Proof. Suppose that $P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=P_{\alpha_M}-\lim_{\overline{x}\to\overline{p}}g(\overline{x})=\overline{L}$, where $\alpha_M=\max\{\alpha_1,\alpha_2\}$. By the Resolution principle, for $\alpha_M\leq\alpha\leq1$, for $\varepsilon>0$, there exist $\delta_1,\delta_2>0$ such that

$$0<\left\|\left(\left|x_{1,\alpha}-p_{1,\alpha}\right|,\left|x_{2,\alpha}-p_{2,\alpha}\right|\right)\right\|<\delta_1$$

implies

$$\left\|\left(\left|f_1(x_{1,\alpha},x_{2,\alpha})-L_{2,\alpha}\right|,\left|f_2(x_{1,\alpha},x_{2,\alpha})-L_{1,\alpha}\right|\right)\right\|<\varepsilon$$

and

$$0<\left\|\left(\left|x_{1,\alpha}-p_{2,\alpha}\right|,\left|x_{2,\alpha}-p_{1,\alpha}\right|\right)\right\|<\delta_2$$

implies

$$\left\|\left(\left|g_1(x_{1,\alpha},x_{2,\alpha})-L_{2,\alpha}\right|,\left|g_2(x_{1,\alpha},x_{2,\alpha})-L_{1,\alpha}\right|\right)\right\|<\varepsilon.$$

Since $f(\overline{x}) \leq h(\overline{x}) \leq g(\overline{x})$ for all $\overline{x} \in \overline{\overline{I}} \backslash \overline{a}$, then, for $\alpha_M \leq \alpha \leq 1$, for there exists $\delta_3 > 0$ such that

$$0 < \left\| \left(\left| x_{1,\alpha} - p_{2,\alpha} \right|, \left| x_{2,\alpha} - p_{1,\alpha} \right| \right) \right\| < \delta_3$$

implies

$$\begin{aligned} \left(L_{1,\alpha} - \frac{\varepsilon}{\sqrt{2}}, L_{2,\alpha} - \frac{\varepsilon}{\sqrt{2}} \right) &< \left(f_1(x_{1,\alpha}, x_{2,\alpha}), f_2(x_{1,\alpha}, x_{2,\alpha}) \right) \\ &\leq \left(h_1(x_{1,\alpha}, x_{2,\alpha}), h_2(x_{1,\alpha}, x_{2,\alpha}) \right) \\ &\leq \left(g_1(x_{1,\alpha}, x_{2,\alpha}), g_2(x_{1,\alpha}, x_{2,\alpha}) \right) \\ &< \left(L_{1,\alpha} + \frac{\varepsilon}{\sqrt{2}}, L_{2,\alpha} + \frac{\varepsilon}{\sqrt{2}} \right). \end{aligned}$$

Choosing $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, we get

$$0 < \left\| \left(\left| x_{1,\alpha} - p_{2,\alpha} \right|_1, \left| x_{2,\alpha} - p_{1,\alpha} \right|_2 \right) \right\| < \delta$$

implies

$$\left(\left| h_1(x_{1,\alpha}, x_{2,\alpha}) - L_{2,\alpha} \right|, \left| h_2(x_{1,\alpha}, x_{2,\alpha}) - L_{1,\alpha} \right| \right) < \left(\frac{\varepsilon}{\sqrt{2}}, \frac{\varepsilon}{\sqrt{2}} \right),$$

which completes the proof.

Corollary 2.21. Suppose $\overline{p} \in \overline{\overline{I}} \subset \overline{\overline{R}}$, where $\overline{\overline{I}}$ is an open fuzzy interval, and f, g, h are fuzzy functions such that $f(\overline{x}) \leq h(\overline{x}) \leq g(\overline{x})$ for all $\overline{x} \in \overline{\overline{I}} \backslash \overline{p}$. We have the following

1. If $P_{\alpha_1} - \lim\limits_{\overline{x} \to \overline{p}} f(\overline{x})$ and $T - \lim\limits_{\overline{x} \to \overline{p}} g(\overline{x})$ are exist, then
$$P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} f(\overline{x}) = P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} h(\overline{x}) = P_{\alpha_1} - \lim_{\overline{x} \to \overline{p}} g(\overline{x}).$$
2. If $T - \lim\limits_{\overline{x} \to \overline{p}} f(\overline{x})$ and $P_{\alpha_2} - \lim\limits_{\overline{x} \to \overline{p}} g(\overline{x})$ are exist, then
$$P_{\alpha_2} - \lim_{\overline{x} \to \overline{p}} f(\overline{x}) = P_{\alpha_2} - \lim_{\overline{x} \to \overline{p}} h(\overline{x}) = P_{\alpha_2} - \lim_{\overline{x} \to \overline{p}} g(\overline{x}).$$
3. If $T - \lim\limits_{\overline{x} \to \overline{p}} f(\overline{x})$ and $T - \lim\limits_{\overline{x} \to \overline{p}} g(\overline{x})$ are exist, then
$$T - \lim_{\overline{x} \to \overline{p}} f(\overline{x}) = T - \lim_{\overline{x} \to \overline{p}} h(\overline{x}) = T - \lim_{\overline{x} \to \overline{p}} g(\overline{x}).$$

3. ONE-SIDED FUZZY LIMIT

Herein we establish the concept of the one-side fuzzy limit of fuzzy functions through the following theorems, the proofs of which are similar to those mentioned in the theorems in Section 2.

Theorem 3.1. Let $f: \overline{\overline{I}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function defined on some open fuzzy interval $\overline{\overline{I}}$ with left endpoint $\overline{p}$ and $\overline{L} \in \overline{\overline{\mathbb{R}}}$. If, for all $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, the α-cuts' boundaries of $f(\overline{x})$ converge to the α-cuts' boundaries of $\overline{L}$ as the α-cuts' boundaries of $\overline{x}$ approach the α-cuts' boundaries of $\overline{p}$, then $f(\overline{x})$ converges partially to $\overline{L}$ as $\overline{x}$ approaches $\overline{p}$ from the right.

Corollary 3.2. Let $f: \overline{\overline{I}} \subset \overline{\overline{\overline{\mathbb{R}}}} \longrightarrow \overline{\overline{\overline{\mathbb{R}}}}$ be a fuzzy function defined on some open fuzzy interval $\overline{\overline{I}}$ with left endpoint $\overline{p}$ and $\overline{L} \in \overline{\overline{\overline{\mathbb{R}}}}$. If, for all $\alpha \in (0,1]$, the α-cuts' boundaries of $f(\overline{x})$ converge to the α-cuts' boundaries of $\overline{L}$ as the α-cuts' boundaries of $\overline{x}$ approach the α-cuts' boundaries of $\overline{p}$, then $f(\overline{x})$ converges totally to $\overline{L}$ as $\overline{x}$ approaches $\overline{p}$ from the right.

Theorem 3.3. Let $f: \overline{\overline{I}} \subset \overline{\overline{\overline{\mathbb{R}}}} \longrightarrow \overline{\overline{\overline{\mathbb{R}}}}$ be a fuzzy function defined on some open fuzzy interval $\overline{\overline{I}}$ with left endpoint $\overline{p}$ and $\overline{L} \in \overline{\overline{\overline{\mathbb{R}}}}$. Then $f(\overline{x})$ converges partially to $\overline{L}$ as $\overline{x}$ approaches $\overline{p}$ from the right iff for all $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$(0,0) < \left(x_{1,\alpha} - p_{2,\alpha}, x_{2,\alpha} - p_{1,\alpha}\right) < (\delta_1, \delta_2)$$

implies (3.1)

$$\left\|\left(\left|f_1\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{2,\alpha}\right|, \left|f_2\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{1,\alpha}\right|\right)\right\| < \varepsilon.$$

Corollary 3.4. Let $f: \overline{\overline{I}} \subset \overline{\overline{\overline{\mathbb{R}}}} \longrightarrow \overline{\overline{\overline{\mathbb{R}}}}$ be a fuzzy function defined on some open fuzzy interval $\overline{\overline{I}}$ with left endpoint $\overline{p}$ and $\overline{L} \in \overline{\overline{\overline{\mathbb{R}}}}$. Then $f(\overline{x})$ converges totally to $\overline{L}$ as $\overline{x}$ approaches $\overline{p}$ from the right iff for all $\alpha \in (0,1]$, for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$(0,0) < \left(x_{1,\alpha} - p_{2,\alpha}, x_{2,\alpha} - p_{1,\alpha}\right) < (\delta_1, \delta_2)$$

implies

$$\left\|\left(\left|f_1\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{2,\alpha}\right|, \left|f_2\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{1,\alpha}\right|\right)\right\| < \varepsilon.$$

Theorem 3.5. Let $f: \overline{\overline{I}} \subset \overline{\overline{\overline{\mathbb{R}}}} \longrightarrow \overline{\overline{\overline{\mathbb{R}}}}$ be a fuzzy function defined on some open fuzzy interval $\overline{\overline{I}}$ with right endpoint $\overline{p}$ and $\overline{L} \in \overline{\overline{\overline{\mathbb{R}}}}$. If, for all $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, the α-cuts' boundaries of $f(\overline{x})$ converge to the α-cuts' boundaries of $\overline{L}$ as the α-cuts' boundaries of $\overline{x}$ approach the α-cuts' boundaries of $\overline{p}$, then $f(\overline{x})$ converges partially to $\overline{L}$ as $\overline{x}$ approaches $\overline{p}$ from the left.

Corollary 3.6. Let $f: \overline{\overline{I}} \subset \overline{\overline{\overline{\mathbb{R}}}} \longrightarrow \overline{\overline{\overline{\mathbb{R}}}}$ be a fuzzy function defined on some open fuzzy interval $\overline{\overline{I}}$ with right endpoint $\overline{p}$ and $\overline{L} \in \overline{\overline{\overline{\mathbb{R}}}}$. If, for all $\alpha \in (0,1]$, the α-cuts' boundaries of $f(\overline{x})$ converge to the α-cuts' boundaries of $\overline{L}$ as the α-cuts' boundaries of $\overline{x}$ approach the α-cuts' boundaries of $\overline{p}$, then $f(\overline{x})$ converges totally to $\overline{L}$ as $\overline{x}$ approaches $\overline{p}$ from the left.

Theorem 3.7. Let $f: \overline{\overline{I}} \subset \overline{\overline{\overline{\mathbb{R}}}} \longrightarrow \overline{\overline{\overline{\mathbb{R}}}}$ be a fuzzy function defined on some open fuzzy interval $\overline{\overline{I}}$ with right endpoint $\overline{p}$ and $\overline{L} \in \overline{\overline{\overline{\mathbb{R}}}}$. Then $f(\overline{x})$ converges partially to $\overline{L}$ as $\overline{x}$ approaches $\overline{p}$ from the left iff for all $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$, for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$(-\delta_1, -\delta_2) < \left(x_{1,\alpha} - p_{2,\alpha}, x_{2,\alpha} - p_{1,\alpha}\right) < (0,0)$$

implies (3.2)

$$\left\|\left(\left|f_1\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{2,\alpha}\right|, \left|f_2\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{1,\alpha}\right|\right)\right\| < \varepsilon.$$

Corollary 3.8. Let $f:\bar{\bar{I}} \subset \bar{\bar{\mathbb{R}}} \longrightarrow \bar{\bar{\mathbb{R}}}$ be a fuzzy function defined on some open fuzzy interval $\bar{\bar{I}}$ with right endpoint $\bar{p}$ and $\bar{L} \in \bar{\bar{\mathbb{R}}}$. Then $f(\bar{x})$ converges totally to $\bar{L}$ as $\bar{x}$ approaches $\bar{p}$ from the left iff for all $\alpha \in (0,1]$, for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$(-\delta_1, -\delta_2) < \left(x_{1,\alpha} - p_{2,\alpha}, x_{2,\alpha} - p_{1,\alpha}\right) < (0,0)$$

implies

$$\left\| \left(\left| f_1\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{2,\alpha} \right|, \left| f_2\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{1,\alpha} \right| \right) \right\| < \varepsilon.$$

Remarks 3.9.

1. We call the convergence in Theorem 3.3 by the partial right-hand fuzzy convergence and $\bar{L}$ by the partial right-hand fuzzy limit of f at $\bar{p}$ and write it as

$$f\left(\bar{p}^{+}\right) = \bar{L} = P_{\alpha_1} - \lim_{\bar{x} \to \bar{p}^{+}} f(\bar{x}) \tag{3.3}$$

because the limits of α-cuts do not exist for $0 < \alpha \leq \alpha_o$. We also call the convergence in corollary 3.4 by the total fuzzy convergence and $\bar{L}$ by the total fuzzy limit of f at $\bar{p}$ and write it as

$$f\left(\bar{p}^{+}\right) = \bar{L} = T - \lim_{\bar{x} \to \bar{p}^{+}} f(\bar{x}) \tag{3.4}$$

because the limits of α-cuts exist for all $\alpha \in (0,1]$. Moreover, if the right-hand fuzzy convergence does not exist for all $\alpha \in (0,1]$, we denote $P_0 - \lim_{\bar{x} \to \bar{p}^{+}} f(\bar{x})$.

2. We call the convergence in Theorem 3.7 by the partial left-hand fuzzy convergence and $\bar{L}$ by the partial left-hand fuzzy limit of f at $\bar{p}$ and write it as

$$f(\bar{p}^{-}) = \bar{L} = P_{\alpha_1} - \lim_{\bar{x} \to \bar{p}^{-}} f(\bar{x}) \tag{3.5}$$

because the limits of α-cut do not exist for $0 < \alpha \leq \alpha_o$. We also call the convergence in Corollary 3.8 by the total fuzzy convergence and $\bar{L}$ by the total fuzzy limit of f at $\bar{p}$ and write it as

$$f(\bar{p}^{-}) = \bar{L} = T - \lim_{\bar{x} \to \bar{p}^{-}} f(\bar{x}) \tag{3.6}$$

because the limits of α-cuts exist for all $\alpha \in (0,1]$. Furthermore, if the right-hand fuzzy convergence does not exist for all $\alpha \in (0,1]$, we denote $P_0 - \lim_{\bar{x} \to \bar{p}^{-}} f(\bar{x})$.

Examples 3.10.

1. Let $f(\bar{x}) = \frac{\bar{1}}{\bar{x} - \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right)}$ and $\bar{p} = \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right)$. Then $P_0 - \lim_{\bar{x} \to \bar{p}^{+}} f(\bar{x})$ and $P_0 - \lim_{\bar{x} \to \bar{p}^{-}} f(\bar{x})$, since,

by the Resolution Principle, for all $\alpha \in (0,1]$, for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$(0,0) < \left(x_{1,\alpha} - \left(-\frac{1}{6}\alpha + \frac{1}{2}\right), x_{2,\alpha} - \left(\frac{1}{12}\alpha + \frac{1}{4}\right)\right) < (\delta_1, \delta_2)$$

implies

$$\left\|\left(\left|\frac{1}{x_{2,\alpha} - \left(\frac{1}{12}\alpha + \frac{1}{4}\right)} - \infty\right|, \left|\frac{1}{x_{1,\alpha} - \left(-\frac{1}{6}\alpha + \frac{1}{4}\right)} - \infty\right|\right)\right\| > \left\|\left(\frac{1}{\delta_2}, \frac{1}{\delta_1}\right)\right\| > \varepsilon$$

and for all $\varepsilon' > 0$, there exist $\delta_1', \delta_2' > 0$ such that

$$(-\delta_1', -\delta_2') < \left(x_{1,\alpha} - \left(-\frac{1}{6}\alpha + \frac{1}{2}\right), x_{2,\alpha} - \left(\frac{1}{12}\alpha + \frac{1}{4}\right)\right) < (0,0)$$

implies

$$\left\|\left(\left|\frac{1}{x_{2,\alpha} - \left(\frac{1}{12}\alpha + \frac{1}{4}\right)} - \infty\right|, \left|\frac{1}{x_{1,\alpha} - \left(-\frac{1}{6}\alpha + \frac{1}{4}\right)} - \infty\right|\right)\right\| > \left\|\left(\frac{1}{\delta_2'}, \frac{1}{\delta_1'}\right)\right\| > \varepsilon'.$$

2. Let $f(\overline{x}) = \exp\left(\frac{\overline{1}}{\overline{x}}\right)$ and $\overline{p} = \overline{0}$. Then, $P_0 - \lim_{\overline{x} \to \overline{p}^+} f(\overline{x}) = \infty$ and $T - \lim_{\overline{x} \to \overline{p}^+} f(\overline{x}) = \overline{0}$, because by the Resolution Principle, for all $\alpha \in (0,1]$, for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$(0,0) < \left(x_{1,\alpha}, x_{2,\alpha}\right) < (\delta_1, \delta_2)$$

implies

$$\left\|\left(\left|\exp\left(\frac{1}{x_{2,\alpha}}\right) - \infty\right|, \left|\exp\left(\frac{1}{x_{1,\alpha}}\right) - \infty\right|\right)\right\| > \left\|\left(\left|\exp\left(\frac{1}{\delta_2}\right) - \infty\right|, \left|\exp\left(\frac{1}{\delta_1}\right) - \infty\right|\right)\right\| > \varepsilon$$

and for all $\varepsilon' > 0$, there exist $\delta_1', \delta_2' > 0$ such that

$$(-\delta_1', -\delta_2') < \left(x_{1,\alpha}, x_{2,\alpha}\right) < (0,0)$$

implies

$$\left\|\left(\left|\exp\left(\frac{1}{x_{2,\alpha}}\right)\right|, \left|\exp\left(\frac{1}{x_{1,\alpha}}\right)\right|\right)\right\| < \left\|\left(\left|\exp\left(-\frac{1}{\delta_2'}\right)\right|, \left|\exp\left(-\frac{1}{\delta_1'}\right)\right|\right)\right\| < \varepsilon'.$$

3. Let $f(\overline{x}) = \begin{cases} \overline{x}^2 & , \overline{x} < \left(\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right) \\ \left(\frac{1}{36}, \frac{1}{25}, \frac{1}{16}\right) & , \left(\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right) < \overline{x} \end{cases}$ and $\overline{p} = \left(\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right)$.

Then $T - \lim_{\overline{x} \to \overline{p}^-} f(\overline{x}) = \left(\frac{1}{36}, \frac{1}{25}, \frac{1}{16}\right)$, because by the Resolution Principle, for all $\alpha \in (0,1]$, for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$(0,0) < \left(x_{1,\alpha} - \left(\frac{-1}{20}\alpha + \frac{1}{4}\right), x_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right) < (\delta_1, \delta_2)$$

implies

$$\left\| \left(\left| \left(\frac{11}{900}\alpha + \frac{1}{36}\right) - \left(\frac{11}{900}\alpha + \frac{1}{36}\right) \right|, \left| \left(\frac{-9}{400}\alpha + \frac{1}{16}\right) - \left(\frac{-9}{400}\alpha + \frac{1}{16}\right) \right| \right) \right\| < \varepsilon$$

and for $\overline{x}$ approaches $\overline{p}$ from the right we have three cases:

Case 1: $T - \lim_{\overline{x} \to \overline{p}^+} f(\overline{x}) = \left(\frac{1}{36}, \frac{1}{25}, \frac{1}{16}\right)$ because by the Resolution principle, for all $\alpha \in (0,1]$, for all $\varepsilon' > 0$, there exist $\delta_1', \delta_2' > 0$ such that

$$(-\delta_1', -\delta_2') < \left(x_{1,\alpha} - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right), x_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right) < (0,0)$$

implies

$$\left\| \left(\left| y_{1,\alpha} - \left(\frac{-1}{20}\alpha + \frac{1}{4}\right)^2 \right|, \left| y_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2 \right| \right) \right\| < \varepsilon',$$

where $y_{1,\alpha} = \min\{x_{1,\alpha}^2, x_{2,\alpha}^2\}, y_{2,\alpha} = \max\{x_{1,\alpha}^2, x_{2,\alpha}^2\}$ and

$$\left| x_{1,\alpha}^2 - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2 \right| \le \left| x_{1,\alpha} - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right) \right| \left| x_{1,\alpha} + \left(-\frac{1}{20}\alpha + \frac{1}{4}\right) \right|$$

$$< \left(|x_{1,\alpha}| + 2\left(-\frac{1}{20}\alpha + \frac{1}{4}\right) \right) \delta_1' < 4\left(-\frac{1}{20}\alpha + \frac{1}{4}\right) \delta_1',$$

$$\left| x_{2,\alpha}^2 - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2 \right| \le \left| x_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right) \right| \left| x_{2,\alpha} + \left(\frac{1}{30}\alpha + \frac{1}{6}\right) \right|$$

$$< \left(|x_{2,\alpha}| + 2\left(\frac{1}{30}\alpha + \frac{1}{6}\right) \right) \delta_2' < 4\left(\frac{1}{30}\alpha + \frac{1}{6}\right) \delta_2',$$

$$\left| x_{1,\alpha}^2 - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2 \right| \le \left| x_{1,\alpha} - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right) \right| \left| x_{1,\alpha} + \left(-\frac{1}{20}\alpha + \frac{1}{4}\right) \right| + \left| \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2 - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2 \right|$$

$$< 4\left(-\frac{1}{20}\alpha + \frac{1}{4}\right) \delta_1' + \left| \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2 - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2 \right|$$

$$< 5\left(-\frac{1}{20}\alpha + \frac{1}{4}\right) \delta_1',$$

$$\left| x_{2,\alpha}^2 - \left(\frac{-1}{20}\alpha + \frac{1}{4}\right)^2 \right| \le \left| x_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right) \right| \left| x_{2,\alpha} + \left(\frac{1}{30}\alpha + \frac{1}{6}\right) \right| + \left| \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2 - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2 \right|$$

$$< 4\left(\frac{1}{30}\alpha + \frac{1}{6}\right)\delta_2' + \left|\left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2 - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2\right|$$
$$< 5\left(\frac{1}{30}\alpha + \frac{1}{6}\right)\delta_2'.$$

Case 2: $P_1 - \lim_{\overline{x}\to\overline{p}^+} f(\overline{x}) = \left(\frac{1}{36}, \frac{1}{25}, \frac{1}{16}\right)$ because by the Resolution Principle, for all $\alpha = 1$, for all $\varepsilon' > 0$, there exist $\delta_1', \delta_2' > 0$ such that

$$(-\delta_1', -\delta_2') < \left(x_{1,\alpha} - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right), x_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right) < (0,0)$$

implies

$$\left\|\left(\left|y_{1,\alpha} - \left(\frac{-1}{20}\alpha + \frac{1}{4}\right)^2\right|, \left|y_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2\right|\right)\right\| < \varepsilon',$$

where $y_{1,\alpha} = x_{2,\alpha}^2, y_{2,\alpha} = x_{1,\alpha}^2$ and

$$\left|x_{2,\alpha}^2 - \left(\frac{-1}{20}\alpha + \frac{1}{4}\right)^2\right| \leq \left|x_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right|\left|x_{2,\alpha} + \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right| + \left|\left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2 - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2\right|$$
$$< 4\left(\frac{1}{30}\alpha + \frac{1}{6}\right)\delta_2',$$
$$\left|x_{1,\alpha}^2 - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2\right| \leq \left|x_{1,\alpha} - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)\right|\left|x_{1,\alpha} + \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)\right| + \left|\left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2 - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2\right|$$
$$< 4\left(-\frac{1}{20}\alpha + \frac{1}{4}\right)\delta_1'.$$

Case 3: $P_0 - \lim_{\overline{x}\to\overline{p}^+} f(\overline{x})$ if $y_{1,\alpha} = y_{2,\alpha} = x_{1,\alpha}x_{2,\alpha}$ for all $\alpha \in (0,1]$, where

$$\left|x_{1,\alpha}x_{2,\alpha} - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2\right|$$
$$\leq \left[\left|x_{1,\alpha} - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)\right| + \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)\right]\left[\left|x_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right| + \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right] + \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2$$
$$< \left[\delta_1 + \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)\right]\left[\delta_2 + \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right] + \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)^2,$$

$$\left|x_{1,\alpha}x_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2\right|$$
$$\leq \left[\left|x_{1,\alpha} - \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)\right| + \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)\right]\left[\left|x_{2,\alpha} - \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right| + \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right] + \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2$$

$$< \left[\delta_1 + \left(-\frac{1}{20}\alpha + \frac{1}{4}\right)\right]\left[\delta_2 + \left(\frac{1}{30}\alpha + \frac{1}{6}\right)\right] + \left(\frac{1}{30}\alpha + \frac{1}{6}\right)^2.$$

Theorem 3.11. Let $f: \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function, $\overline{L} \in \overline{\overline{\mathbb{R}}}$ and $P_{\alpha_1} - \lim_{\overline{x}\to\overline{p}^-} f(\overline{x}), P_{\alpha_2} - \lim_{\overline{x}\to\overline{p}^+} f(\overline{x})$ exist. Then $P_{\alpha_M} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = \overline{L}$ iff $\overline{L} = P_{\alpha_M} - \lim_{\overline{x}\to\overline{p}^-} f(\overline{x}) = P_{\alpha_M} - \lim_{\overline{x}\to\overline{p}^+} f(\overline{x}), \alpha_M = \max\{\alpha_1, \alpha_2\}$.

Proof. Suppose that $P_{\alpha_M} - \lim_{\overline{x}\to\overline{p}} f(\overline{x}) = \overline{L}$, where $\alpha_M = \max\{\alpha_1, \alpha_2\}$. By the Resolution Principle, for $0 < \alpha_0 < \alpha_M \leq \alpha \leq 1$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \left\|\left(\left|x_{1,\alpha} - p_{2,\alpha}\right|, \left|x_{2,\alpha} - p_{1,\alpha}\right|\right)\right\| < \delta$$

implies

$$\left\|\left(\left|f_1\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{2,\alpha}\right|, \left|f_2\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{1,\alpha}\right|\right)\right\| < \varepsilon.$$

Since, for $0 < \alpha_0 < \alpha_M \leq \alpha \leq 1$, that

$$(0,0) < \left(x_{1,\alpha} - p_{2,\alpha}, x_{2,\alpha} - p_{1,\alpha}\right) < (\delta_1, \delta_2)$$

and

$$(-\delta_1, -\delta_2) < \left(x_{1,\alpha} - p_{2,\alpha}, x_{2,\alpha} - p_{1,\alpha}\right) < (0,0)$$

imply

$$0 < \left\|\left(\left|x_{1,\alpha} - p_{2,\alpha}\right|, \left|x_{2,\alpha} - p_{1,\alpha}\right|\right)\right\| < \delta,$$

then

$$(0,0) < \left(x_{1,\alpha} - p_{2,\alpha}, x_{2,\alpha} - p_{1,\alpha}\right) < \left(\frac{\delta_1}{\sqrt{2}}, \frac{\delta_2}{\sqrt{2}}\right)$$

implies

$$\left\|\left(\left|f_1\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{2,\alpha}\right|, \left|f_2\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{1,\alpha}\right|\right)\right\| < \varepsilon$$

and

$$\left(-\frac{\delta_1}{\sqrt{2}}, -\frac{\delta_2}{\sqrt{2}}\right) < \left(x_{1,\alpha} - p_{2,\alpha}, x_{2,\alpha} - p_{1,\alpha}\right) < (0,0)$$

implies

$$\left\|\left(\left|f_1\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{2,\alpha}\right|, \left|f_2\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{1,\alpha}\right|\right)\right\| < \varepsilon.$$

Conversely, suppose $\overline{L} = P_{\alpha_M} - \lim_{\overline{x}\to\overline{p}-} f(\overline{x}) = P_{\alpha_M} - \lim_{\overline{x}\to\overline{p}+} f(\overline{x})$ holds. By the Resolution principle, for all $0 < \alpha_0 < \alpha_M \leq \alpha \leq 1$, for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$(0,0) < \left(x_{1,\alpha} - p_{2,\alpha}, x_{2,\alpha} - p_{1,\alpha}\right) < \left(\frac{\delta_1}{\sqrt{2}}, \frac{\delta_2}{\sqrt{2}}\right)$$

implies

$$\left\|\left(\left|f_1\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{2,\alpha}\right|, \left|f_2\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{1,\alpha}\right|\right)\right\| < \varepsilon$$

and

$$\left(-\frac{\delta_1}{\sqrt{2}},-\frac{\delta_2}{\sqrt{2}}\right) < \left(x_{1,\alpha}-p_{2,\alpha},x_{2,\alpha}-p_{1,\alpha}\right) < (0,0)$$

implies

$$\left\|\left(\left|f_1\left(x_{1,\alpha},x_{2,\alpha}\right)-L_{2,\alpha}\right|,\left|f_2\left(x_{1,\alpha},x_{2,\alpha}\right)-L_{1,\alpha}\right|\right)\right\| < \varepsilon.$$

Set $\delta=\min\{\delta_1,\delta_2\}$. Then

$$0 < \left\|\left(\left|x_{1,\alpha}-p_{2,\alpha}\right|_1,\left|x_{2,\alpha}-p_{1,\alpha}\right|_2\right)\right\| < \delta$$

implies

$$\left\|\left(\left|f\left(x_{1,\alpha},x_{2,\alpha}\right)-L_{2,\alpha}\right|,\left|f_2\left(x_{1,\alpha},x_{2,\alpha}\right)-L_{1,\alpha}\right|\right)\right\| < \varepsilon.$$

Corollary 3.12. Let $f:\overline{\overline{\mathbb{R}}}\to\overline{\overline{\mathbb{R}}}$ be a fuzzy function and $\overline{L}\in\overline{\overline{\mathbb{R}}}$. We have the following

1. If $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^-}f(\overline{x}),P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^+}f(\overline{x})$ exist. Then $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=\overline{L}$ iff $\overline{L}=P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^-}f(\overline{x})=P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^+}f(\overline{x})$.
2. If $T-\lim_{\overline{x}\to\overline{p}^-}f(\overline{x}),T-\lim_{\overline{x}\to\overline{p}^+}f(\overline{x})$ exist. Then $T-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=\overline{L}$ iff $\overline{L}=T-\lim_{\overline{x}\to\overline{p}^-}f(\overline{x})=T-\lim_{\overline{x}\to\overline{p}^+}f(\overline{x})$.
3. If $T-\lim_{\overline{x}\to\overline{p}^-}f(\overline{x}),P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^+}f(\overline{x})$ exist. Then $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=\overline{L}$ iff $\overline{L}=P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^-}f(\overline{x})=P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^+}f(\overline{x})$.
4. If $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^-}f(\overline{x}),T-\lim_{\overline{x}\to\overline{p}^+}f(\overline{x})$ exist. Then $P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=\overline{L}$ iff $\overline{L}=P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^-}f(\overline{x})=P_{\alpha_1}-\lim_{\overline{x}\to\overline{p}^+}f(\overline{x})$.

Examples 3.13.

1. Let $f(\overline{x})=\frac{|\sin(\overline{x})|}{\sin(\overline{x})}$ and $\overline{p}=\overline{0}$. Then $P_0-\lim_{\overline{x}\to\overline{p}}f(\overline{x})$ because by the Resolution Principle, for all $\alpha\in(0,1]$, $\lim_{\substack{x_{1,\alpha}\to0^+\\x_{2,\alpha}\to0^+}}\min\left\{\frac{|\sin(x_{i,\alpha})|}{\sin(x_{i,\alpha})}:i=1,2\right\},\lim_{\substack{x_{1,\alpha}\to0^+\\x_{2,\alpha}\to0^+}}\max\left\{\frac{|\sin(x_{i,\alpha})|}{\sin(x_{i,\alpha})}:i=1,2\right\}$ give positive values and $\lim_{\substack{x_{1,\alpha}\to0^-\\x_{2,\alpha}\to0^-}}\min\left\{\frac{|\sin(x_{i,\alpha})|}{\sin(x_{i,\alpha})}:i=1,2\right\},\lim_{\substack{x_{1,\alpha}\to0^-\\x_{2,\alpha}\to0^-}}\max\left\{\frac{|\sin(x_{i,\alpha})|}{\sin(x_{i,\alpha})}:i=1,2\right\}$ give negative values.

2. Let $f(\overline{x})=\begin{cases}\overline{2}\overline{x}+\overline{1} & ,\overline{x} > \overline{1}\\ \overline{5} & ,\overline{x} = \overline{1}\\ \overline{7}\overline{x}^2-\overline{4} & ,\overline{x} < \overline{1}\end{cases}$ and $\overline{x}=\overline{1}$. Then $T-\lim_{\overline{x}\to\overline{p}}f(\overline{x})=\overline{3}$ because by

the Resolution Principle, for all $\alpha \in (0,1]$, we have the α-cuts

$$\left[\lim_{\substack{x_{1,\alpha}\to 1^+\\ x_{2,\alpha}\to 1^+}}\left(2x_{1,\alpha}+1\right), \lim_{\substack{x_{1,\alpha}\to 1^+\\ x_{2,\alpha}\to 1^+}}\left(2x_{2,\alpha}+1\right)\right] = [3,3],$$

$$\left[\lim_{\substack{x_{1,\alpha}\to 1^-\\ x_{2,\alpha}\to 1^-}}\left(7y_{1,\alpha}-4\right), \lim_{\substack{x_{1,\alpha}\to 1^-\\ x_{2,\alpha}\to 1^-}}\left(7y_{2,\alpha}-4\right)\right] = [3,3].$$

where $y_{1,\alpha} = \min\{x_{i,\alpha}x_{j,\alpha}: i,j = 1,2\}, y_{2,\alpha} = \max\{x_{i,\alpha}x_{j,\alpha}: i,j = 1,2\}$. Thus, $\lim_{\overline{x}\to\overline{1}^+} f(\overline{x}) = \overline{3}$ and $\lim_{\overline{x}\to\overline{1}^-} f(\overline{x}) = \overline{3}$.

4. FUZZY LIMIT AT INFINITY AND INFINITY FUZZY LIMIT

Concepts of the fuzzy limit of fuzzy function at infinity and infinity fuzzy limit with their properties are obtained in this section through the following theorems.

Theorem 4.1. Let $f: \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function, $\overline{L} \in \overline{\overline{\mathbb{R}}}$ and $(\overline{a}, \infty) \subseteq \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. If, for all $0 < \alpha_o < \alpha_1 \le \alpha \le 1$, the α-cuts' boundaries of $f(\overline{x})$ converge to the α-cuts' boundaries of $\overline{L}$ as the α-cuts' boundaries of $\overline{x}$ approach ∞, then $f(\overline{x})$ converges partially to $\overline{L}$ as $\overline{x}$ approaches ∞.

Corollary 4.2. Let $f: \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function, $\overline{L} \in \overline{\overline{\mathbb{R}}}$ and $(\overline{a}, \infty) \subseteq \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. If, for all $\alpha \in (0,1]$, the α-cuts' boundaries of $f(\overline{x})$ converge to the α-cuts' boundaries of $\overline{L}$ as the α-cuts' boundaries of $\overline{x}$ approach ∞, then $f(\overline{x})$ converges totally to $\overline{L}$ as $\overline{x}$ approaches ∞.

Theorem 4.3. Let $f: \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function, $\overline{L} \in \overline{\overline{\mathbb{R}}}$ and $(\overline{a}, \infty) \subseteq \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ converges partially to $\overline{L}$ as $\overline{x}$ approaches ∞ iff for all $0 < \alpha_0 < \alpha_1 \le \alpha \le 1$, for all $\varepsilon > 0$, there exists $\overline{K}$ such that the α-cuts $\left[K_{1,\alpha}, K_{2,\alpha}\right]$ of $\overline{K}$ satisfy $K_{1,\alpha} = K_{1,\alpha}(\varepsilon) > a_{1,\alpha}, K_{2,\alpha} = K_{2,\alpha}(\varepsilon) > a_{2,\alpha}$ and $\left(x_{1,\alpha}, x_{2,\alpha}\right) > \left(K_{1,\alpha}, K_{2,\alpha}\right)$ implies

$$\left\|\left(\left|f_1\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{2,\alpha}\right|, \left|f_2\left(x_{1,\alpha}, x_{2,\alpha}\right) - L_{1,\alpha}\right|\right)\right\| < \varepsilon. \tag{4.1}$$

Corollary 4.4. Let $f: \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function, $\overline{L} \in \overline{\overline{\mathbb{R}}}$ and $(\overline{a}, \infty) \subseteq \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ converges totally to $\overline{L}$ as $\overline{x}$ approaches ∞ iff for all $\alpha \in (0,1]$, for all $\varepsilon > 0$, there exists $\overline{K}$ such that the α-cuts $\left[K_{1,\alpha}, K_{2,\alpha}\right]$ of $\overline{K}$ satisfy $K_{1,\alpha} = K_{1,\alpha}(\varepsilon) > a_{1,\alpha}, K_{2,\alpha} = K_{2,\alpha}(\varepsilon) > a_{2,\alpha}$ and $\left(x_{1,\alpha}, x_{2,\alpha}\right) > \left(K_{1,\alpha}, K_{2,\alpha}\right)$ implies

$$\left\|\left(\left|f_1(x_{1,\alpha},x_{2,\alpha})-L_{2,\alpha}\right|,\left|f_2(x_{1,\alpha},x_{2,\alpha})-L_{1,\alpha}\right|\right)\right\|<\varepsilon.$$

Theorem 4.5. Let $f:\overline{\overline{E}}\subset\overline{\overline{\mathbb{R}}}\longrightarrow\overline{\overline{\mathbb{R}}}$ be a fuzzy function, $\overline{L}\in\overline{\overline{\mathbb{R}}}$ and $(-\infty,\overline{a})\subseteq\overline{\overline{E}}$ for some $\overline{a}\in\overline{\overline{\mathbb{R}}}$. If, for all $0<\alpha_o<\alpha_1\leq\alpha\leq1$, the α-cuts' boundaries of $f(\overline{x})$ converge to the α-cuts' boundaries of $\overline{L}$ as the α-cuts' boundaries of $\overline{x}$ approach $-\infty$, then $f(\overline{x})$ converges partially to $\overline{L}$ as $\overline{x}$ approaches $-\infty$.

Corollary 4.6. Let $f:\overline{\overline{E}}\subset\overline{\overline{\mathbb{R}}}\longrightarrow\overline{\overline{\mathbb{R}}}$ be a fuzzy function, $\overline{L}\in\overline{\overline{\mathbb{R}}}$ and $(-\infty,\overline{a})\subseteq\overline{\overline{E}}$ for some $\overline{a}\in\overline{\overline{\mathbb{R}}}$. If, for all $\alpha\in(0,1]$, the α-cuts' boundaries of $f(\overline{x})$ converge to the α-cuts' boundaries of $\overline{L}$ as the α-cuts' boundaries of $\overline{x}$ approach $-\infty$, then $f(\overline{x})$ converges totally to $\overline{L}$ as $\overline{x}$ approaches $-\infty$.

Theorem 4.7. Let $f:\overline{\overline{E}}\subset\overline{\overline{\mathbb{R}}}\longrightarrow\overline{\overline{\mathbb{R}}}$ be a fuzzy function, $\overline{L}\in\overline{\overline{\mathbb{R}}}$ and $(-\infty,\overline{a})\subseteq\overline{\overline{E}}$ for some $\overline{a}\in\overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ converges partially to $\overline{L}$ as $\overline{x}$ approaches $-\infty$ iff for all $0<\alpha_o<\alpha_1\leq\alpha\leq1$, for all $\varepsilon>0$, there exists $\overline{K}$ such that the α-cuts $[K_{1,\alpha},K_{2,\alpha}]$ of $\overline{K}$ satisfy $K_{1,\alpha}=K_{1,\alpha}(\varepsilon)<a_{1,\alpha},K_{2,\alpha}=K_{2,\alpha}(\varepsilon)<a_{2,\alpha}$ and $(x_{1,\alpha},x_{2,\alpha})<(K_{1,\alpha},K_{2,\alpha})$ implies

$$\left\|\left(\left|f_1(x_{1,\alpha},x_{2,\alpha})-L_{2,\alpha}\right|,\left|f_2(x_{1,\alpha},x_{2,\alpha})-L_{1,\alpha}\right|\right)\right\|<\varepsilon. \quad (4.2)$$

Corollary 4.8. Let $f:\overline{\overline{E}}\subset\overline{\overline{\mathbb{R}}}\rightarrow\overline{\overline{\mathbb{R}}}$ be a fuzzy function, $\overline{L}\in\overline{\overline{\mathbb{R}}}$ and $(-\infty,\overline{a})\subseteq\overline{\overline{E}}$ for some $\overline{a}\in\overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ converges totally to $\overline{L}$ as $\overline{x}$ approaches $-\infty$ iff for all $\alpha\in(0,1]$, for all $\varepsilon>0$, there exists $\overline{K}$ such that the α-cuts $[K_{1,\alpha},K_{2,\alpha}]$ of $\overline{K}$ satisfy $K_{1,\alpha}=K_{1,\alpha}(\varepsilon)<a_{1,\alpha},K_{2,\alpha}=K_{2,\alpha}(\varepsilon)<a_{2,\alpha}$ and $(x_{1,\alpha},x_{2,\alpha})<(K_{1,\alpha},K_{2,\alpha})$ implies

$$\left\|\left(\left|f_1(x_{1,\alpha},x_{2,\alpha})-L_{2,\alpha}\right|,\left|f_2(x_{1,\alpha},x_{2,\alpha})-L_{1,\alpha}\right|\right)\right\|<\varepsilon.$$

Remarks 4.9.

1. The fuzzy convergence in Theorem 4.3 is denoted by

$$f(\infty)=\overline{L}=P_{\alpha_1}-\lim_{\overline{x}\to\infty}f(\overline{x}) \quad (4.3)$$

because the limits of α-cuts do not exist for $0<\alpha\leq\alpha_o$. The fuzzy convergence in Corollary 4.4 is denoted by

$$f(\infty)=\overline{L}=T-\lim_{\overline{x}\to\infty}f(\overline{x}) \quad (4.4)$$

because the limits of α-cuts exist for all $\alpha\in(0,1]$. Moreover, if the fuzzy convergence does not exist for all $\alpha\in(0,1]$, we write $P_0-\lim_{\overline{x}\to\infty}f(\overline{x})$.

2. The fuzzy convergence in Theorem 4.7 is denoted by

$$f(-\infty)=\overline{L}=P_{\alpha_1}-\lim_{\overline{x}\to-\infty}f(\overline{x}) \quad (4.5)$$

because the limits of α-cuts do not exist for $0 < \alpha \le \alpha_o$. The fuzzy convergence in Corollary 4.8 is denoted by

$$f(-\infty) = \overline{L} = T - \lim_{\overline{x} \to -\infty} f(\overline{x}) \qquad (4.6)$$

because the limits of α-cuts exist for all $\alpha \in (0,1]$. Furthermore, if the fuzzy convergence does not exist for all $\alpha \in (0,1]$, we write $P_0 - \lim_{\overline{x} \to -\infty} f(\overline{x})$.

Examples 4.10.

1. $T - \lim_{\overline{x} \to \infty} \frac{\overline{2}\overline{x}^2 - \overline{1}}{\overline{1} - \overline{x}^2} = -\overline{2} = T - \lim_{\overline{x} \to -\infty} \frac{\overline{2}\overline{x}^2 - \overline{1}}{\overline{1} - \overline{x}^2}$ because by Resolution Principle, for all $\alpha \in (0,1]$, the α-cut

$$\left[\frac{[2,2][x_{1,\alpha}, x_{2,\alpha}]^2 - [1,1]}{[1,1] - [x_{1,\alpha}, x_{2,\alpha}]^2}\right] = \left[\min_{i,j=1,2}\left\{\frac{2x_{i,\alpha}x_{j,\alpha} - 1}{1 - x_{i,\alpha}x_{j,\alpha}}\right\}, \max_{i,j=1,2}\left\{\frac{2x_{i,\alpha}x_{j,\alpha} - 1}{1 - x_{i,\alpha}x_{j,\alpha}}\right\}\right]$$

of $\frac{\overline{2}\overline{x}^2 - \overline{1}}{\overline{1} - \overline{x}^2}$ has the limit

$$\left[\lim_{\substack{x_{1,\alpha} \to \pm\infty \\ x_{2,\alpha} \to \pm\infty}} \min_{i,j=1,2}\left\{\frac{2x_{i,\alpha}x_{j,\alpha} - 1}{1 - x_{i,\alpha}x_{j,\alpha}}\right\}, \lim_{\substack{x_{1,\alpha} \to \pm\infty \\ x_{2,\alpha} \to \pm\infty}} \max_{i,j=1,2}\left\{\frac{2x_{i,\alpha}x_{j,\alpha} - 1}{1 - x_{i,\alpha}x_{j,\alpha}}\right\}\right] =$$

$$\left[\lim_{\substack{x_{1,\alpha} \to \pm\infty \\ x_{2,\alpha} \to \pm\infty}} \min_{i,j=1,2}\left\{\frac{2 - \frac{1}{x_{i,\alpha}x_{j,\alpha}}}{-1 + \frac{1}{x_{i,\alpha}x_{j,\alpha}}}\right\}, \lim_{\substack{x_{1,\alpha} \to \pm\infty \\ x_{2,\alpha} \to \pm\infty}} \max_{i,j=1,2}\left\{\frac{2 - \frac{1}{x_{i,\alpha}x_{j,\alpha}}}{-1 + \frac{1}{x_{i,\alpha}x_{j,\alpha}}}\right\}\right] = [-2, -2].$$

2. $T - \lim_{\overline{x} \to \infty} \frac{\overline{1}}{\overline{x}} = \overline{0} = T - \lim_{\overline{x} \to -\infty} \frac{\overline{1}}{\overline{x}}$ because by Resolution Principle, for all $\alpha \in (0,1]$, for all $\varepsilon > 0$, there exist α-cuts $[K_{1,\alpha}, K_{2,\alpha}]$ of $\overline{K} > \overline{0}$ such that $(x_{1,\alpha}, x_{2,\alpha}) > (K_{1,\alpha}, K_{2,\alpha})$ implies

$$\left\|\left(\left|\frac{1}{x_{2,\alpha}}\right|, \left|\frac{1}{x_{1,\alpha}}\right|\right)\right\| < \left\|\left(\left|\frac{1}{K_{2,\alpha}}\right|, \left|\frac{1}{K_{1,\alpha}}\right|\right)\right\| < \varepsilon$$

and for all $\varepsilon' > 0$, there exist α-cuts $[K'_{1,\alpha}, K'_{2,\alpha}]$ of $\overline{K'} > \overline{0}$ such that $(x_{1,\alpha}, x_{2,\alpha}) < (-K'_{1,\alpha}, -K'_{2,\alpha})$ implies

$$\left\|\left(\left|\frac{1}{x_{2,\alpha}}\right|, \left|\frac{1}{x_{1,\alpha}}\right|\right)\right\| < \left\|\left(\left|\frac{1}{K'_{2,\alpha}}\right|, \left|\frac{1}{K'_{1,\alpha}}\right|\right)\right\| < \varepsilon'.$$

Theorem 4.11. Let $f: \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function and $\overline{\overline{I}}$ be an open fuzzy interval such that $\overline{\overline{I}} \backslash \overline{p} \subset \overline{\overline{E}}$ for some $\overline{p} \in \overline{\overline{\mathbb{R}}}$. If for all $\alpha \in (0,1]$, the α-cuts' boundaries of $f(\overline{x})$ tend to ∞ as the α-cuts' boundaries of $\overline{x}$ approach the α-cuts' boundaries of $\overline{p}$, then $f(\overline{x})$ tends to ∞ as $\overline{x}$ approaches $\overline{p}$.

Theorem 4.12. Let $f: \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function and $\overline{\overline{I}}$ be an open fuzzy interval such that $\overline{\overline{I}} \backslash \overline{p} \subset \overline{\overline{E}}$ for some $\overline{p} \in \overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ tends to ∞ as $\overline{x}$ approaches $\overline{p}$ iff, for all $\alpha \in (0,1]$, for all $\overline{\beta} > 0$, there exists $\delta > 0$ such that

$$0 < \|(|x_{1,\alpha} - p_{2,\alpha}|, |x_{2,\alpha} - p_{1,\alpha}|)\| < \delta$$

implies (4.7)

$$\left(f_1(x_{1,\alpha}, x_{2,\alpha}), f_2(x_{1,\alpha}, x_{2,\alpha})\right) > (\beta_{1,\alpha}, \beta_{2,\alpha}).$$

Theorem 4.13. Let $f: \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function and $\overline{\overline{I}}$ be an open fuzzy interval such that $\overline{\overline{I}} \backslash \overline{a} \subset \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. If for all $\alpha \in (0,1]$, the α-cuts' boundaries of $f(\overline{x})$ tend to $-\infty$ as the α-cuts' boundaries of $\overline{x}$ approach the α-cuts' boundaries of $\overline{a}$, then $f(\overline{x})$ tends to $-\infty$ as $\overline{x}$ approaches $\overline{a}$.

Theorem 4.14. Let $f: \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function and $\overline{\overline{I}}$ be an open fuzzy interval such that $\overline{\overline{I}} \backslash \overline{p} \subset \overline{\overline{E}}$ for some $\overline{p} \in \overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ tends to $-\infty$ as $\overline{x}$ approaches $\overline{p}$ iff, for all $\alpha \in (0,1]$, for all $\overline{\beta} > 0$, there exists $\delta > 0$ such that

$$0 < \|(|x_{1,\alpha} - p_{2,\alpha}|, |x_{2,\alpha} - p_{1,\alpha}|)\| < \delta$$

implies (4.8)

$$\left(f_1(x_{1,\alpha}, x_{2,\alpha}), f_2(x_{1,\alpha}, x_{2,\alpha})\right) < (\beta_{1,\alpha}, \beta_{2,\alpha}).$$

Remark 4.15. The fuzzy convergence in Theorem 4.12 is denoted as

$$T - \lim_{\overline{x} \to \overline{p}} f(\overline{x}) = \infty \qquad (4.9)$$

and the fuzzy convergence in theorem 4.14 is denoted as

$$T - \lim_{\overline{x} \to \overline{p}} f(\overline{x}) = -\infty. \qquad (4.10)$$

Examples 4.16.

1. $\lim_{\overline{x} \to \overline{0}} \frac{1}{\overline{x}^2} = \infty$ because by the Resolution Principle, for all $\alpha \in (0,1]$, there exist α-cuts $[M_{1,\alpha}, M_{2,\alpha}]$ of $\overline{M} \in \overline{\overline{\mathbb{R}}}$ such that

$$0 < \| (x_{1,\alpha}, x_{2,\alpha}) \| < \delta_1 \text{ implies } f_1(x_{1,\alpha}, x_{2,\alpha}) > \frac{1}{\delta_1^2} \Rightarrow \delta_1 = \frac{1}{M_{2,\alpha}},$$
$$0 < \| (x_{1,\alpha}, x_{2,\alpha}) \| < \delta_2 \text{ implies } f_2(x_{1,\alpha}, x_{2,\alpha}) > \frac{1}{\delta_2^2} \Rightarrow \delta_2 = \frac{1}{M_{1,\alpha}},$$

where

$$f_1(x_{1,\alpha},x_{2,\alpha}) = \min\left\{\frac{1}{x_{1,\alpha}^2},\frac{1}{x_{1,\alpha}x_{2,\alpha}},\frac{1}{x_{2,\alpha}^2}\right\},$$
$$f_2(x_{1,\alpha},x_{2,\alpha}) = \max\left\{\frac{1}{x_{1,\alpha}^2},\frac{1}{x_{1,\alpha}x_{2,\alpha}},\frac{1}{x_{2,\alpha}^2}\right\}.$$

2. $\lim_{\overline{x}\to\overline{1}-}\frac{\overline{x}+\overline{2}}{\overline{2}\overline{x}^2-\overline{3}\overline{x}+\overline{1}}=-\infty$ because by the Resolution Principle, for all $\alpha\in(0,1]$, there exist α-cuts $[M_{1,\alpha},M_{2,\alpha}]$ of $\overline{M}<\overline{0}$ such that

$$0<\left\|\left|x_{1,\alpha}-1\right|,\left|x_{2,\alpha}-1\right|\right\|<\delta_1$$

implies

$$f_1(x_{1,\alpha},x_{2,\alpha}) = \min_{i,j=1,2}\left\{\frac{x_{i,\alpha}+2}{2x_{i,\alpha}x_{j,\alpha}-3x_{i,\alpha}+1}\right\}<M_{1,\alpha}$$

and

$$0<\left\|\left|x_{1,\alpha}-1\right|,\left|x_{2,\alpha}-1\right|\right\|<\delta_2$$

implies

$$f_2(x_{1,\alpha},x_{2,\alpha}) = \max_{i,j=1,2}\left\{\frac{x_{i,\alpha}+2}{2x_{i,\alpha}x_{j,\alpha}-3x_{i,\alpha}+1}\right\}<M_{2,\alpha},$$

where $2x_{i,\alpha}x_{j,\alpha}-3x_{i,\alpha}+1$ is negative and converges to 0 as $(x_{1,\alpha},x_{2,\alpha})$ approaches to $(1,1)$ from the left. Therefore, choosing $\delta_i\in(0,1), i=1,2$ such that $(1-\delta_1,1-\delta_1)<(x_{1,\alpha},x_{2,\alpha})<(1,1)$ and $(1-\delta_2,1-\delta_2)<(x_{1,\alpha},x_{2,\alpha})<(1,1)$ imply $\frac{2}{M_1}<2x_{i,\alpha}x_{j,\alpha}-3x_{i,\alpha}+1$ and $\frac{2}{M_2}<2x_{i,\alpha}x_{j,\alpha}-3x_{i,\alpha}+1$ respectively. Since $(0,0)<(x_{1,\alpha},x_{2,\alpha})<(1,1)$ implies $(2,2)<(x_{1,\alpha}+2,x_{2,\alpha}+2)<(3,3)$, we obtain the result.

Theorem 4.17. Let $f:\overline{\overline{E}}\subset\overline{\overline{\mathbb{R}}}\longrightarrow\overline{\overline{\mathbb{R}}}$ be a fuzzy function and $(\overline{a},\infty)\subseteq\overline{\overline{E}}$ for some $\overline{a}\in\overline{\overline{\mathbb{R}}}$. If, for all $\alpha\in(0,1]$, the α-cuts' boundaries of $f(\overline{x})$ tend to ∞ as the α-cuts' boundaries of $\overline{x}$ approach ∞, then $f(\overline{x})$ tends to ∞ as $\overline{x}$ approaches ∞.

Theorem 4.18. Let $f:\overline{\overline{E}}\subset\overline{\overline{\mathbb{R}}}\longrightarrow\overline{\overline{\mathbb{R}}}$ be a fuzzy function and $(\overline{a},\infty)\subseteq\overline{\overline{E}}$ for some $\overline{a}\in\overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ tends to ∞ as $\overline{x}$ approaches ∞ iff for all $\alpha\in(0,1]$, for all $\overline{\beta}>0$, there exists $\overline{K}$ such that the α-cuts $[K_{1,\alpha},K_{2,\alpha}]$ of $\overline{K}$ satisfy $K_{1,\alpha}=K_{1,\alpha}(\varepsilon)>a_{1,\alpha}, K_{2,\alpha}=K_{2,\alpha}(\varepsilon)>a_{2,\alpha}$ and $(x_{1,\alpha},x_{2,\alpha})>(K_{1,\alpha},K_{2,\alpha})$ implies

$$\left(f_1(x_{1,\alpha},x_{2,\alpha}),f_2(x_{1,\alpha},x_{2,\alpha})\right)>(\beta_{1,\alpha},\beta_{2,\alpha}). \tag{4.11}$$

Theorem 4.19. Let $f:\overline{\overline{E}}\subset\overline{\overline{\mathbb{R}}}\longrightarrow\overline{\overline{\mathbb{R}}}$ be a fuzzy function and $(\overline{a},\infty)\subseteq\overline{\overline{E}}$ for some $\overline{a}\in\overline{\overline{\mathbb{R}}}$. If, for all $\alpha\in(0,1]$, α-cuts' boundaries of $f(\overline{x})$ tend to $-\infty$ as the α-cuts' boundaries of $\overline{x}$ approach ∞, then $f(\overline{x})$ tends to $-\infty$ as $\overline{x}$ approaches ∞.

Theorem 4.20. Let $f : \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function and $(\overline{a}, \infty) \subseteq \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ tends to $-\infty$ as $\overline{x}$ approaches ∞ iff for all $\alpha \in (0,1]$, for all $\overline{\beta} > 0$, there exists $\overline{K}$ such that the α-cuts $[K_{1,\alpha}, K_{2,\alpha}]$ of $\overline{K}$ satisfy $K_{1,\alpha} = K_{1,\alpha}(\varepsilon) > a_{1,\alpha}, K_{2,\alpha} = K_{2,\alpha}(\varepsilon) > a_{2,\alpha}$ and $(x_{1,\alpha}, x_{2,\alpha}) > (K_{1,\alpha}, K_{2,\alpha})$ implies

$$\left(f_1(x_{1,\alpha}, x_{2,\alpha}), f_2(x_{1,\alpha}, x_{2,\alpha})\right) < (\beta_{1,\alpha}, \beta_{2,\alpha}). \tag{4.12}$$

Remark 4.21. The fuzzy convergence in Theorem 4.18 is denoted as

$$T - \lim_{\overline{x} \to \infty} f(\overline{x}) = \infty \tag{4.13}$$

and the fuzzy convergence in Theorem 4.20 is denoted as

$$T - \lim_{\overline{x} \to \infty} f(\overline{x}) = -\infty. \tag{4.14}$$

Theorem 4.22. Let $f : \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function and $(-\infty, \overline{a}) \subseteq \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. If, for all $\alpha \in (0,1]$, the α-cuts' boundaries of $f(\overline{x})$ tend to $-\infty$ as the α-cuts' boundaries of $\overline{x}$ approach $-\infty$, then $f(\overline{x})$ tends to $-\infty$ as $\overline{x}$ approaches $-\infty$.

Theorem 4.23. Let $f : \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function and $(-\infty, \overline{a}) \subseteq \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ tends to $-\infty$ as $\overline{x}$ approaches $-\infty$ iff for all $\alpha \in (0,1]$, for all $\varepsilon > 0$, there exists $\overline{K}$ such that the α-cuts $[K_{1,\alpha}, K_{2,\alpha}]$ of $\overline{K}$ satisfy $K_{1,\alpha} = K_{1,\alpha}(\varepsilon) < a_{1,\alpha}, K_{2,\alpha} = K_{2,\alpha}(\varepsilon) < a_{2,\alpha}$ and $(x_{1,\alpha}, x_{2,\alpha}) < (K_{1,\alpha}, K_{2,\alpha})$ implies

$$\left(f_1(x_{1,\alpha}, x_{2,\alpha}), f_2(x_{1,\alpha}, x_{2,\alpha})\right) < (\beta_{1,\alpha}, \beta_{2,\alpha}). \tag{4.15}$$

Theorem 4.24. Let $f : \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function and $(-\infty, \overline{a}) \subseteq \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. If, for all $\alpha \in (0,1]$, the α-cuts' boundaries of $f(\overline{x})$ tend to ∞ as the α-cuts' boundaries of $\overline{x}$ approach $-\infty$, then $f(\overline{x})$ tends to ∞ as $\overline{x}$ approaches $-\infty$.

Theorem 4.25. Let $f : \overline{\overline{E}} \subset \overline{\overline{\mathbb{R}}} \longrightarrow \overline{\overline{\mathbb{R}}}$ be a fuzzy function and $(-\infty, \overline{a}) \subseteq \overline{\overline{E}}$ for some $\overline{a} \in \overline{\overline{\mathbb{R}}}$. Then $f(\overline{x})$ tends to ∞ as $\overline{x}$ approaches $-\infty$ iff for all $\alpha \in (0,1]$, for all $\varepsilon > 0$, there exists $\overline{K}$ such that the α-cuts $[K_{1,\alpha}, K_{2,\alpha}]$ of $\overline{K}$ satisfy $K_{1,\alpha} = K_{1,\alpha}(\varepsilon) < a_{1,\alpha}, K_{2,\alpha} = K_{2,\alpha}(\varepsilon) < a_{2,\alpha}$ and $(x_{1,\alpha}, x_{2,\alpha}) < (K_{1,\alpha}, K_{2,\alpha})$ implies

$$\left(f_1(x_{1,\alpha}, x_{2,\alpha}), f_2(x_{1,\alpha}, x_{2,\alpha})\right) > (\beta_{1,\alpha}, \beta_{2,\alpha}). \tag{4.16}$$

Remark 4.26. The fuzzy convergence in Theorem 4.23 is denoted as

$$=T - \lim_{\overline{x} \to -\infty} f(\overline{x}) = -\infty \tag{4.17}$$

and the fuzzy convergence in Theorem 4.25 is denoted as

$$T - \lim_{\overline{x} \to -\infty} f(\overline{x}) = \infty. \tag{4.18}$$

5. CONCLUSION

The theory of fuzzy limits of fuzzy functions has been estalished according to Altai's principle, and by applying the Representation Theorem (i.e. Resolution Principle) to run fuzzy arithmetics. Moreover, the convergence of the fuzzy function to its fuzzy limit has been proven depending on the convergence of its α-cuts' boundaries to their limits for the membership degree $0 < \alpha_o < \alpha_1 \leq \alpha \leq 1$.

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