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DEGREE EXPONENT SUM ENERGY OF COMMUTING GRAPH FOR DIHEDRAL GROUPS

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Abstract: For a finite group G and a nonempty subset X of G, we construct a graph with a set of vertex X such that any pair of distinct vertices of X are adjacent if they are commuting elements in G. This graph is known as the commuting graph of G on X, denoted by $\Gamma_G[X]$. The degree exponent sum (DES) matrix of a graph is a square matrix whose (p,q)-th entry is $d_{v_p}{}^{d_{v_q}} + d_{v_q}{}^{d_{v_p}}$ whenever p is different from q, otherwise, it is zero, where d_{v_p} (or d_{v_q}) is the degree of the vertex v_p (or vertex, v_q) of a graph. This study presents results for the DES energy of commuting graph for dihedral groups of order $2n$, using the absolute eigenvalues of its DES matrix.

Keywords: *Commuting graph, dihedral group, degree exponent sum matrix, the energy of a graph.*

1. Introduction

A group is a set of elements associated by a binary operation, which satisfies closure property, has a unique identity element, and unique inverses for each element in the group (Aschbacher, 2000). Suppose now that $\boldsymbol{G}$ is any finite group and $\boldsymbol{Z(G)}$ is the center of $\boldsymbol{G}$. The commuting graph of $\boldsymbol{G}$ on a nonempty subset $\boldsymbol{X}$ of $\boldsymbol{G}$, denoted by $\boldsymbol{\Gamma_G[X]}$, is a graph whose vertex set is $\boldsymbol{X}$, and two distinct vertices are adjacent if they commute in $\boldsymbol{G}$. If $\boldsymbol{X = G\backslash Z(G)}$, then we write $\boldsymbol{\Gamma_G := \Gamma_G[X]}$ and $\boldsymbol{\Gamma_G}$ is called the commuting graph of $\boldsymbol{G}$. This graph is a simple undirected graph introduced by Brauer and Fowler (1955).

The commuting graph of $\boldsymbol{G}$ on $\boldsymbol{X}$ has been further associated with the spectral graph theory, where matrices are associated with a graph. The adjacency matrix $\boldsymbol{A(\Gamma_G[X]) = [a_{pq}]}$ of $\boldsymbol{\Gamma_G[X]}$, is an $\boldsymbol{n \times n}$ matrix, defined by its entries $\boldsymbol{a_{pq}}$ are equal to 1 if there is an edge between the vertices $\boldsymbol{v_p}$, $\boldsymbol{v_q}$, and 0 otherwise. Clearly, $\boldsymbol{A(\Gamma_G[X])}$ is a symmetric matrix with zero diagonal entries since $\boldsymbol{\Gamma_G[X]}$ is a simple graph. For real numbers $\boldsymbol{\lambda}$ and an $\boldsymbol{n \times n}$ identity matrix $\boldsymbol{I_n}$, the characteristic polynomial of $\boldsymbol{\Gamma_G[X]}$ is defined by $\boldsymbol{P_{A(\Gamma_G[X])}(\lambda) = \det(\lambda I_n - A(\Gamma_G[X]))}$. The roots of $\boldsymbol{P_{A(\Gamma_G[X])}(\lambda) = 0}$ are $\boldsymbol{\lambda_1, \lambda_2, \dots, \lambda_n}$ and are known as the eigenvalues of $\boldsymbol{\Gamma_G[X]}$.

By the definition of adjacency matrix, the (ordinary) spectrum of the finite graph $\boldsymbol{\Gamma_G[X]}$ is the list of eigenvalues $\boldsymbol{\lambda_1, \lambda_2, \dots, \lambda_m}$, with their respective multiplicities $\boldsymbol{k_1, k_2, \dots, k_m}$ as exponents, denoted by $\boldsymbol{Spec(\Gamma_G[X]) = \left\{\lambda_1^{(k_1)}, \lambda_2^{(k_2)}, \dots, \lambda_m^{(k_m)}\right\}}$. Furthermore, the energy of $\boldsymbol{\Gamma_G[X]}$ is the sum of the absolute eigenvalues of $\boldsymbol{A(\Gamma_G[X])}$, which is $\boldsymbol{E(\Gamma_G[X]) = \sum_{i=1}^{n} |\lambda_i|}$. Other than that, Gutman found this definition in 1978 by considering a chemical molecule as a graph and estimating the $\boldsymbol{\pi}$-electron energy.

Several studies regarding the commuting graph involve the spectrum and energy of its adjacency matrix. For finite non-abelian groups, Dutta and Nath (2017a) and Dutta and Nath (2017b) have described the formula for the spectrum of the commuting graph. Laplacian spectrum, signless Laplacian spectrum and their corresponding energies of the commuting graph of dihedral groups can be found in Dutta and Nath (2018) and Dutta and Nath (2021). Furthermore, the discussion of the adjacency energy for the subgroup graph of the dihedral group has been done by Abdussakir *et al.* (2019). In 2022, Sharafdini *et al.* discussed the commuting graph for some finite groups with abelian centralizers and found the energy for some particular families of AC groups.

Apart from the adjacency matrix, Laplacian matrix, and signless Laplacian matrix, another matrix related to the degree of vertices in a graph defined by Basavanagoud and Eshwarachandra in 2020 is the principal focus point here, called the degree exponent sum (DES) matrix. A limited number of studies central to the DES matrices for the commuting graph have been found. This fact motivates us to have a detailed description of the DES energy for the commuting graphs of $\boldsymbol{G}$.

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In this paper, we focus on $\Gamma_G[X]$ constructed on the non-abelian dihedral group of order $2n, n \geq 3$, denoted as $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$. The center of D_{2n}, $Z(D_{2n})$ is either $\{e\}$ **if n is odd** or $\{e, a^{\frac{n}{2}}\}$ **if n is even**. The centralizer of the element a^i in the group D_{2n} is $C_{D_{2n}}(a^i) = \{a^i : 1 \leq i \leq n\}$ and for the element $a^i b$ is either $C_{D_{2n}}(a^i b) = \{e, a^i b\}$, if n is odd or $C_{D_{2n}}(a^i b) = \{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\}$, **if n is even**.

2. Preliminaries

Now, we are ready to see the definition of the degree exponent sum (DES) matrix, considering d_{v_p} as the degree of v_p, which is the number of vertices adjacent to v_p. Moreover, if every vertex has the same degree r, then the graph is called r-regular graph.

Definition 2.1. (Basavanagoud & Eshwarachandra, 2020) The DES matrix of order $n \times n$ associated with $\Gamma_G[X]$ is given by $DES(\Gamma_G[X]) = [des_{pq}]$ whose (p, q)-th entry is

$$des_{pq} = \begin{cases} d_{v_p}{}^{d_{v_q}} + d_{v_q}{}^{d_{v_p}}, & \text{if } p \neq q \\ 0, & \text{if } p = q \end{cases}.$$

Therefore, the DES energy of $\Gamma_G[X]$ can be defined as follows:

$$E_{DES}(\Gamma_G[X]) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues (not necessarily distinct) of $DES(\Gamma_G[X])$.

In this section, we include some previous results beneficial for the next section. The following lemma is important for computing the characteristic polynomial of the commuting graph Γ_G.

Lemma 2.1: (Ramane & Shinde, 2017) If a, b, c and d are real numbers, and J_n is an $n \times n$ matrix whose all elements are equal to 1, then the determinant of the $(n_1 + n_2) \times (n_1 + n_2)$ matrix of the form

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix},$$

can be simplified as given in the following expression
$(\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}\big((\lambda - (n_1 - 1)a)(\lambda - (n_2 - 1)b) - n_1 n_2 cd\big)$,
where $1 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.

A graph with n vertices, where every vertex is adjacent to all other vertices, is called a complete graph K_n and the complement of K_n is denoted by $\bar{K}_n$. The following lemma is the result of the spectrum of K_n, which is useful in computing $E_{DES}(\Gamma_G[X])$.

Lemma 2.2: (Brouwer & Haemers, 2010) If K_n is the complete graph on n vertices, then its adjacency matrix is $J_n - I_n$ and the spectrum of K_n is $\{(n-1)^{(1)}, (-1)^{(n-1)}\}$.

3. Main Results

This section presents several results on the degree exponent sum (DES) energy of the commuting graph on the dihedral group of order $2n$. We divide n into two cases, namely when n is odd and n is even. This is strictly for $n \geq 3$ since the dihedral group is abelian for $n = 1$ and $n = 2$.

Recall that the dihedral group of order $2n$ is $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$. Let the set of rotation elements of D_{2n}, which are not members of $Z(D_{2n})$, be written as $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z(D_{2n})$ and $G_2 = \{a^i b : 1 \leq i \leq n\}$ be the set of reflection elements of D_{2n}. The following is the result of the degree of each vertex in the commuting graph of D_{2n}.

Theorem 3.1: Let $\Gamma_{D_{2n}}$ be the commuting graph of D_{2n}. Then,

1. the degree of a^i in $\Gamma_{D_{2n}}$, denoted as d_{a^i}, is given by
$$d_{a^i} = \begin{cases} n-2, & \text{if } n \text{ is odd} \\ n-3, & \text{if } n \text{ is even} \end{cases},$$
2. the degree of $a^i b$ in $\Gamma_{D_{2n}}$, denoted as $d_{a^i b}$, is given by
$$d_{a^i b} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}.$$

Proof.

1. If n is odd, then $Z(D_{2n}) = \{e\}$. Since $C_{D_{2n}}(a^i) = \{a^i : 1 \leq i \leq n\}$, then $d_{a^i} = n - 2$, removing e and a^i itself. If n is even, then $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$. Consequently, we have $d_{a^i} = n - 3$, removing $e, a^{\frac{n}{2}}$, and a^i itself.
2. If n is odd, the element $a^i b$, where $1 \leq i \leq n$, has the centralizer $C_{D_{2n}}(a^i b) = \{e, a^i b\}$ of size two, then there is no edge between any pair of vertices in Γ_G. Therefore, $d_{a^i b} = 0$. If n is even, the centralizer of each element $a^i b$ is given by
$C_{D_{2n}}(a^i b) = \{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\}$, for all $1 \leq i \leq n$.
Then, by excluding e and $a^{\frac{n}{2}}$, which are the central elements in D_{2n}, there exists only an edge between the vertices $a^i b$ and $a^{\frac{n}{2}+i} b$ in Γ_G, for all $1 \leq i \leq n$. Hence, $d_{a^i b} = 1$.

Consequently, the isomorphism of the commuting graph with the common type of graphs can be seen in the following result:

Theorem 3.2: Let X be any nonempty subset of D_{2n}.

1. If $X = G_1$, then
$\Gamma_{D_{2n}}[X] \cong K_m$, where $m = |G_1|$.

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2. If $X = G_2$, then

$$\Gamma_{D_{2n}}[X] \cong \begin{cases} \bar{K}_n, & \text{if } n \text{ is odd} \\ 1-\text{regular graph}, & \text{if } n \text{ is even} \end{cases}$$

Proof:

1. The centralizer of a^i, for $1 \le i \le n$, is $C_{D_{2n}}(a^i) = \{a^i : 1 \le i \le n\}$ of size n. This implies that every vertex of G_1 is adjacent to all vertices in the set itself. Thus, $\Gamma_{D_{2n}}[G_1] \cong K_m$, where $m = |G_1|$.
2. It follows from Theorem 3.1 that the degree of $a^i b$ in $\Gamma_{D_{2n}}[G_2]$ is all zero for $1 \le i \le n$, where n is odd. Hence, $\Gamma_{D_{2n}}[G_2] \cong \bar{K}_n$, a complement of the complete graph on n vertices. Now, suppose n is even. Again, by Theorem 3.1, the degree of $a^i b$ in $\Gamma_{D_{2n}}[G_2]$ is all 1. This implies that $\Gamma_{D_{2n}}[G_2]$ is disconnected, with each component isomorphic to the 1-regular graph.

We illustrate the two theorems above via the following examples for $n = 4$ and $n = 5$.

Example 1. Let Γ_{D_8} be the commuting graph of D_8, where $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, $Z(D_8) = \{e, a^2\}$, $G_1 = \{a, a^3\}$, $G_2 = \{b, ab, a^2b, a^3b\}$, $C_{D_8}(b) = \{e, a^2, b, a^2b\} = C_{D_8}(a^2b)$, $C_{D_8}(ab) = \{e, a^2, ab, a^3b\} = C_{D_8}(a^3b)$. Using the information on the centralizer of each element in D_8, the commuting graph of D_8 is as in Figure 1.

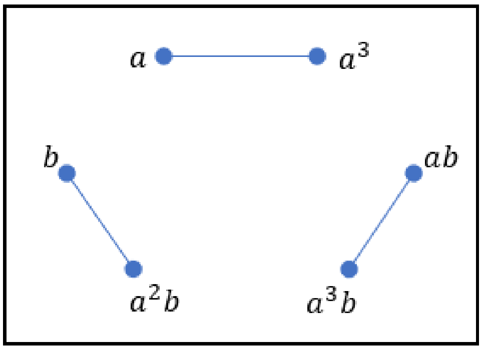

Figure 1. Commuting graph of D_8.

From Figure 1, it is clear that the degree of each vertex a and a^3 is one. In particular, if $X = G_1$, then $\Gamma_{D_8}[G_1]$ is a complete graph on two vertices, K_2. However, for each vertex $a^i b$, for $1 \le i \le 4$, its degree is also one. If $X = G_2$, then $\Gamma_{D_8}[G_2]$ is a disconnected 1-regular graph with two components isomorphic to K_2.

Example 2. Let $\Gamma_{D_{10}}$ be the commuting graph of D_{10}, where $D_{10} = \{e, a, a^2, a^3, a^4 b, ab, a^2b, a^3b, a^4b\}$, $Z(D_{10}) = \{e\}$, $G_1 = \{a, a^2, a^3, a^4\}$, $G_2 = \{b, ab, a^2b, a^3b, a^4b\}$, $C_{D_{10}}(a^ib) = \{a^ib\}$, and $C_{D_{10}}(a^i) = \{a^i : 1 \le i \le 4\}$. Using the information on the centralizer of each element in D_{10}, the commuting graph of D_{10} is as in Figure 2.

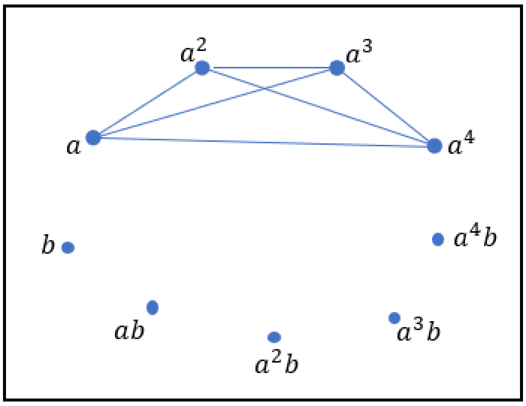

Figure 2. Commuting graph $\Gamma_{D_{10}}$.

From Figure 2, it is clear that the degree of each vertex a^i, where $1 \le i \le 4$ is three. In particular, if $X = G_1$, then $\Gamma_{D_{10}}[G_1]$ is a complete graph on four vertices, K_4. However, for each vertex $a^i b$, for $1 \le i \le 5$, its degree is zero. If $X = G_2$, then $\Gamma_{D_{10}}[G_2]$ is a disconnected graph with five isolated vertices and isomorphic to the complement of a complete graph on five vertices, $\bar{K}_5$.

Theorem 3.3: Let X be any nonempty subset of D_{2n}.

1. If $X = G_1$, then

$$E_{DES}(\Gamma_{D_{2n}}[X]) = \begin{cases} 4(n-2)^{n-1}, & \text{if } n \text{ is odd} \\ 4(n-3)^{n-2}, & \text{if } n \text{ is even} \end{cases}.$$

2. If $X = G_2$, then

$$E_{DES}(\Gamma_{D_{2n}}[X]) = 4(n-1).$$

Proof.

2. **When n is odd**. From Theorem 3.2 (1), $\Gamma_{D_{2n}}[G_1] = K_m$, where $m = |G_1| = n-1$, removing e in $Z(D_{2n})$. Then, every vertex of $\Gamma_{D_{2n}}[G_1]$ has degree $n-2$. Subsequently, we can construct an $(n-1)\times(n-1)$ DES matrix of $\Gamma_{D_{2n}}[G_1]$, $DES(\Gamma_{D_{2n}}[G_1]) = [des_{pq}]$ whose (p,q)-th entry is $des_{pq} = (n-2)^{n-2} + (n-2)^{n-2} = 2(n-2)^{n-2}$, for $p \ne q$, and 0 otherwise:

$$DES(\Gamma_{D_{2n}}[G_1]) = \begin{bmatrix} 0 & 2(n-2)^{n-2} & \cdots & 2(n-2)^{n-2} \\ 2(n-2)^{n-2} & 0 & \cdots & 2(n-2)^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 2(n-2)^{n-2} & 2(n-2)^{n-2} & \cdots & 0 \end{bmatrix}$$

$$= 2(n-2)^{n-2} \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}.$$

In other words, the DES matrix of $\Gamma_{D_{2n}}[G_1]$ is the product of $2(n-2)^{n-2}$ and the adjacency matrix of K_{n-1}. Based

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on Lemma 2.2, $\boldsymbol{Spec(K_{n-1})}$ is given by $\{(\boldsymbol{n}-\boldsymbol{2})^{(1)}, (-\boldsymbol{1})^{(\boldsymbol{n}-2)}\}$. Since the adjacency energy of $\boldsymbol{K_{n-1}}$ is $|\boldsymbol{n}-\boldsymbol{2}| + (\boldsymbol{n}-\boldsymbol{2})|-\boldsymbol{1}| = \boldsymbol{2(n-2)}$, the DES energy of $\boldsymbol{\Gamma_{D_{2n}}[G_1]}$ will be $\boldsymbol{2(n-2)^{n-2}\cdot 2(n-2) = 4(n-2)^{n-1}}$.

When $\boldsymbol{n}$ is even. From Theorem 3.2 (1), $\boldsymbol{\Gamma_{D_{2n}}[G_1] = K_m}$, where $\boldsymbol{m = |G_1| = n-2}$, removing $\boldsymbol{e}$ and $\boldsymbol{a^{\frac{n}{2}}}$ in $\boldsymbol{Z(D_{2n})}$. Then, every vertex of $\boldsymbol{\Gamma_{D_{2n}}[G_1]}$ has degree $\boldsymbol{n-3}$. Consequently, we can construct an $(\boldsymbol{n}-\boldsymbol{2})\times(\boldsymbol{n}-\boldsymbol{2})$ DES matrix of $\boldsymbol{\Gamma_{D_{2n}}[G_1]}$, $\boldsymbol{DES(\Gamma_{D_{2n}}[G_1]) = [des_{pq}]}$ whose $(\boldsymbol{p},\boldsymbol{q})$-th entry is $\boldsymbol{des_{pq} = (n-3)^{n-3} + (n-3)^{n-3} = 2(n-3)^{n-3}}$, for $\boldsymbol{p \neq q}$, and 0 otherwise:

$$\boldsymbol{DES(\Gamma_{D_{2n}}[G_1])} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{2(n-3)^{n-3}} & \cdots & \boldsymbol{2(n-3)^{n-3}} \\ \boldsymbol{2(n-3)^{n-3}} & \boldsymbol{0} & \cdots & \boldsymbol{2(n-3)^{n-3}} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{2(n-3)^{n-3}} & \boldsymbol{2(n-3)^{n-3}} & \cdots & \boldsymbol{0} \end{bmatrix}$$

$$= 2(n-3)^{n-3}\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

Thus, the DES matrix of $\boldsymbol{\Gamma_{D_{2n}}[G_1]}$ is the product of $\boldsymbol{2(n-3)^{n-3}}$ and the adjacency matrix of $\boldsymbol{K_{n-2}}$. Based on Lemma 2.2, $\boldsymbol{Spec(K_{n-2})}$ is given by $\{(\boldsymbol{n}-\boldsymbol{3})^{(1)}, (-\boldsymbol{1})^{(\boldsymbol{n}-3)}\}$. Since the adjacency energy of $\boldsymbol{K_{n-2}}$ is $|\boldsymbol{n}-\boldsymbol{3}| + (\boldsymbol{n}-\boldsymbol{3})|-\boldsymbol{1}| = \boldsymbol{2(n-3)}$, the DES energy of $\boldsymbol{\Gamma_{D_{2n}}[G_1]}$ will be $\boldsymbol{2(n-3)^{n-3}\cdot 2(n-3) = 4(n-3)^{n-2}}$.

2. **When $\boldsymbol{n}$ is odd**. From Theorem 3.2 (2), $\boldsymbol{\Gamma_{D_{2n}}[G_2] = \overline{K}_n}$, where $\boldsymbol{n = |G_2|}$. Then, all of the vertices have degree zero. Correspondingly, we can construct an $\boldsymbol{n\times n}$ DES matrix of $\boldsymbol{\Gamma_{D_{2n}}[G_2]}$, $\boldsymbol{DES(\Gamma_{D_{2n}}[G_2]) = [des_{pq}]}$ whose $(\boldsymbol{p},\boldsymbol{q})$-th entry is $\boldsymbol{des_{pq} = 0^0 + 0^0 = 2}$, for $\boldsymbol{p\neq q}$, and 0 otherwise:

$$\boldsymbol{DES(\Gamma_{D_{2n}}[G_2])} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{2} & \cdots & \boldsymbol{2} \\ \boldsymbol{2} & \boldsymbol{0} & \cdots & \boldsymbol{2} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{2} & \boldsymbol{2} & \cdots & \boldsymbol{0} \end{bmatrix} = 2\begin{bmatrix} \boldsymbol{0} & \boldsymbol{1} & \cdots & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{0} & \cdots & \boldsymbol{1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{1} & \boldsymbol{1} & \cdots & \boldsymbol{0} \end{bmatrix}.$$

In other words, $\boldsymbol{DES(\Gamma_{D_{2n}}[G_2]) = 2A(K_n)}$ is the multiple of two adjacency matrices of $\boldsymbol{K_n}$. Thus, $\boldsymbol{E_{DES}(\Gamma_{D_{2n}}[G_2]) = 2(|n-1| + (n-1)|-1|) = 4(n-1)}$.

When $\boldsymbol{n}$ is even. From Theorem 3.2 (2), $\boldsymbol{\Gamma_{D_{2n}}[G_2]}$ is a regular graph with degree one. Then, we can construct an $\boldsymbol{n\times n}$ DES matrix of $\boldsymbol{\Gamma_{D_{2n}}[G_2]}$, $\boldsymbol{DES(\Gamma_{D_{2n}}[G_2]) = [des_{pq}]}$ whose $(\boldsymbol{p},\boldsymbol{q})$-th entry is $\boldsymbol{des_{pq} = 1^1 + 1^1 = 2}$, for $\boldsymbol{p\neq q}$, and 0 otherwise:

$$\boldsymbol{DES(\Gamma_{D_{2n}}[G_2])} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{2} & \cdots & \boldsymbol{2} \\ \boldsymbol{2} & \boldsymbol{0} & \cdots & \boldsymbol{2} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{2} & \boldsymbol{2} & \cdots & \boldsymbol{0} \end{bmatrix} = 2\begin{bmatrix} \boldsymbol{0} & \boldsymbol{1} & \cdots & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{0} & \cdots & \boldsymbol{1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{1} & \boldsymbol{1} & \cdots & \boldsymbol{0} \end{bmatrix}.$$

It implies that $\boldsymbol{DES(\Gamma_{D_{2n}}[G_2]) = 2A(K_n)}$. Thus, $\boldsymbol{E_{DES}(\Gamma_{D_{2n}}[G_2]) = 4(n-1)}$.

The DES energy of the commuting graph $\boldsymbol{\Gamma_{D_{2n}}[X]}$ for $\boldsymbol{X = G_1, G_2}$ are given by the following examples, for $\boldsymbol{n = 4}$ and $\boldsymbol{n = 5}$.

Example 3. In Figure 1, we have shown the commuting graph of $\boldsymbol{D_8}$. When $\boldsymbol{X = G_1}$, since we only have two vertices $\boldsymbol{a}$ and $\boldsymbol{a^3}$, we have a $\boldsymbol{2\times 2}$ DES matrix of $\boldsymbol{\Gamma_{D_8}[G_1]}$ with the non-diagonal entries are $\boldsymbol{1^1 + 1^1 = 2}$, and the diagonal entries are zero. We then obtain

$$\boldsymbol{DES(\Gamma_{D_8}[G_1])} = \begin{bmatrix} \boldsymbol{0} & 2 \\ 2 & \boldsymbol{0} \end{bmatrix}.$$

Furthermore, the characteristic polynomial of $\boldsymbol{DES(\Gamma_{D_8}[G_1])}$ is $\boldsymbol{P_{DES(\Gamma_{D_8}[G_1])}(\lambda) = \det\left(\lambda I_2 - DES(\Gamma_{D_8}[G_1])\right) = det}\begin{bmatrix} \boldsymbol{\lambda} & \boldsymbol{-2} \\ \boldsymbol{-2} & \boldsymbol{\lambda} \end{bmatrix} = \boldsymbol{\lambda^2 - 4}$. It implies that the eigenvalues of $\boldsymbol{DES(\Gamma_{D_8}[G_1])}$ are $\boldsymbol{\lambda = 2}$ and $\boldsymbol{\lambda = -2}$. Therefore, the DES energy of $\boldsymbol{\Gamma_{D_8}[G_1]}$ is $\boldsymbol{E_{DES}(\Gamma_{D_8}[G_1]) = |2| + |-2| = 4 = 4(4-3)^{4-2}}$.

For the case $\boldsymbol{X = G_2}$, we know that the set of vertices is $\{\boldsymbol{b, ab, a^2b, a^3b}\}$. Here, we have a $\boldsymbol{4\times 4}$ DES matrix of $\boldsymbol{\Gamma_{D_8}[G_2]}$ with the non-diagonal entries are $\boldsymbol{1^1 + 1^1 = 2}$, while the diagonal entries are zero. Then, we get

$$\boldsymbol{DES(\Gamma_{D_8}[G_2])} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} \\ \boldsymbol{2} & \boldsymbol{0} & \boldsymbol{2} & \boldsymbol{2} \\ \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{0} & \boldsymbol{2} \\ \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{0} \end{bmatrix}.$$

Additionally, the characteristic polynomial of $\boldsymbol{DES(\Gamma_{D_8}[G_2])}$ is $\boldsymbol{P_{DES(\Gamma_{D_8}[G_2])}(\lambda) = \det\left(\lambda I_4 - DES(\Gamma_{D_8}[G_2])\right) = (\lambda+2)^3(\lambda-6)}$. It implies that the eigenvalues of $\boldsymbol{DES(\Gamma_{D_8}[G_2])}$ are $\boldsymbol{\lambda = -2}$ with multiplicity 3 and a single $\boldsymbol{\lambda = 6}$. Therefore, $\boldsymbol{E_{DES}(\Gamma_{D_8}[G_2]) = 3|-2| + |6| = 12 = 4(4-1)}$.

Example 4. In Figure 2, we have presented the commuting graph of $\boldsymbol{D_{10}}$. For $\boldsymbol{X = G_1}$, we have a $\boldsymbol{4\times 4}$ DES matrix of $\boldsymbol{\Gamma_{D_{10}}[G_1]}$ with the non-diagonal entries are $\boldsymbol{3^3 + 3^3 = 54}$, while the diagonal entries are zero. We then obtain

$$\boldsymbol{DES(\Gamma_{D_{10}}[G_1])} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{54} & \boldsymbol{54} & \boldsymbol{54} \\ \boldsymbol{54} & \boldsymbol{0} & \boldsymbol{54} & \boldsymbol{54} \\ \boldsymbol{54} & \boldsymbol{54} & \boldsymbol{0} & \boldsymbol{54} \\ \boldsymbol{54} & \boldsymbol{54} & \boldsymbol{54} & \boldsymbol{0} \end{bmatrix}$$

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Furthermore, the characteristic polynomial of $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_1])$ is $\boldsymbol{P}_{\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_1])}(\boldsymbol{\lambda}) = \mathbf{det}\left(\boldsymbol{\lambda I}_4 - \boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_1])\right) = (\boldsymbol{\lambda} + \mathbf{54})^3(\boldsymbol{\lambda} - \mathbf{162})$. It implies that the eigenvalues of $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_1])$ are $\boldsymbol{\lambda} = -\mathbf{54}$ with multiplicity 3 and a single $\boldsymbol{\lambda} = \mathbf{162}$. Therefore, the DES energy of $\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_1]$ is $\boldsymbol{E}_{\boldsymbol{DES}}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_1]) = \mathbf{3}|-\mathbf{54}| + |\mathbf{162}| = \mathbf{324} = \mathbf{4}(\mathbf{5}-\mathbf{2})^{\mathbf{5}-\mathbf{1}}$.

Additionally, for $\boldsymbol{X} = \boldsymbol{G}_2$, we have a $\mathbf{5}\times\mathbf{5}$ DES matrix of $\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_2]$ with the non-diagonal entries are $\mathbf{0}^{\mathbf{0}} + \mathbf{0}^{\mathbf{0}} = \mathbf{2}$, and the diagonal entries are zero. We then obtain

$$\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_2]) = \begin{bmatrix} \mathbf{0} & 2 & 2 & 2 & 2 \\ 2 & \mathbf{0} & 2 & 2 & 2 \\ 2 & 2 & \mathbf{0} & 2 & 2 \\ 2 & 2 & 2 & \mathbf{0} & 2 \\ 2 & 2 & 2 & 2 & \mathbf{0} \end{bmatrix}$$

Hence, the characteristic polynomial of $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_2])$ is $\boldsymbol{P}_{\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_2])}(\boldsymbol{\lambda}) = \mathbf{det}\left(\boldsymbol{\lambda I}_5 - \boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_2])\right) = (\boldsymbol{\lambda} + \mathbf{2})^{\mathbf{4}}(\boldsymbol{\lambda} - \mathbf{8})$. It implies that the eigenvalues of $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_2])$ are $\boldsymbol{\lambda} = -\mathbf{2}$ with multiplicity 4 and $\boldsymbol{\lambda} = \mathbf{8}$ with multiplicity 1. Therefore, $\boldsymbol{E}_{\boldsymbol{DES}}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{10}}[\boldsymbol{G}_2]) = \mathbf{4}|-\mathbf{2}| + |\mathbf{8}| = \mathbf{16} = \mathbf{4}(\mathbf{5}-\mathbf{1})$.

Theorem 3.4: Let $\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}}$ be the commuting graph of $\boldsymbol{D}_{2n}$. Then, the characteristic polynomial of $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}})$ is

1. $P_{DES(\Gamma_{D_{2n}})}(\lambda) = \left(\lambda + 2(n-2)^{(n-2)}\right)^{n-2}(\lambda+2)^{n-1}(\lambda^2 - (2(n-1) + 2(n-2)^{n-1})\lambda + 4(n-2)^{n-1}(n-1) - n(n-1))$, for n is odd, while
2. $P_{DES(\Gamma_{D_{2n}})}(\lambda) = \left(\lambda + 2(n-3)^{(n-3)}\right)^{n-3}(\lambda+2)^{n-1}(\lambda^2 - (2(n-1) + 2(n-3)^{n-2})\lambda + 4(n-1)(n-3)^{n-2} - n(n-2)^3)$, for n is even.

Proof.

1. When $\boldsymbol{n}$ is odd, from Theorem 3.1, we have $\boldsymbol{d}_{\boldsymbol{a}^i} = \boldsymbol{n}-\mathbf{2}$ and $\boldsymbol{d}_{\boldsymbol{a}^i\boldsymbol{b}} = \mathbf{0}$, for all $\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}$. Then, using the fact that $\boldsymbol{Z}(\boldsymbol{D}_{2n}) = \{\boldsymbol{e}\}$, we have $\mathbf{2}\boldsymbol{n}-\mathbf{1}$ vertices in $\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}}$. The set of vertices consists of $\boldsymbol{n}-\mathbf{1}$ vertices of the form $\boldsymbol{a}^{\boldsymbol{i}}$, for $\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}-\mathbf{1}$, and $\boldsymbol{n}$ vertices of the form $\boldsymbol{a}^{\boldsymbol{i}}\boldsymbol{b}$, for $\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}$. Consequently, the DES matrix for $\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}}$ is a $(\mathbf{2}\boldsymbol{n}-\mathbf{1})\times(\mathbf{2}\boldsymbol{n}-\mathbf{1})$ matrix, $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}}) = [\boldsymbol{des}_{\boldsymbol{pq}}]$ whose entries are:
   (i) $\boldsymbol{des}_{\boldsymbol{pq}} = (\boldsymbol{n}-\mathbf{2})^{\boldsymbol{n}-\mathbf{2}} + (\boldsymbol{n}-\mathbf{2})^{\boldsymbol{n}-\mathbf{2}} = \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{\boldsymbol{n}-\mathbf{2}}$, for $\boldsymbol{p} \ne \boldsymbol{q}$, and $\mathbf{1} \le \boldsymbol{p},\boldsymbol{q} \le \boldsymbol{n}-\mathbf{1}$,
   (ii) $\boldsymbol{des}_{\boldsymbol{pq}} = (\boldsymbol{n}-\mathbf{2})^{\mathbf{0}} + (\mathbf{0})^{\boldsymbol{n}-\mathbf{2}} = \mathbf{1}$, for $\mathbf{1} \le \boldsymbol{p} \le \boldsymbol{n}-\mathbf{1}$ and $\boldsymbol{n} \le \boldsymbol{q} \le \mathbf{2}\boldsymbol{n}-\mathbf{1}$,
   (iii) $\boldsymbol{des}_{\boldsymbol{pq}} = (\mathbf{0})^{\boldsymbol{n}-\mathbf{2}} + (\boldsymbol{n}-\mathbf{2})^{\mathbf{0}} = \mathbf{1}$, for $\boldsymbol{n} \le \boldsymbol{p} \le \mathbf{2}\boldsymbol{n}-\mathbf{1}$ and $\mathbf{1} \le \boldsymbol{q} \le \boldsymbol{n}-\mathbf{1}$,
   (iv) $\boldsymbol{des}_{\boldsymbol{pq}} = (\mathbf{0})^{\mathbf{0}} + (\mathbf{0})^{\mathbf{0}} = \mathbf{2}$, for $\boldsymbol{p} \ne \boldsymbol{q}$, $\boldsymbol{n} \le \boldsymbol{p} \le \mathbf{2}\boldsymbol{n}-\mathbf{1}$ and $\boldsymbol{n} \le \boldsymbol{q} \le \mathbf{2}\boldsymbol{n}-\mathbf{1}$,
   (v) $\boldsymbol{des}_{\boldsymbol{pq}} = \mathbf{0}$, for $\boldsymbol{p} = \boldsymbol{q}$.

We can construct $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}})$ as follows:

$$\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}}) = \left[\begin{array}{cccc|cccc} \mathbf{0} & \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})} & \cdots & \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})} & \mathbf{0} & \cdots & \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})} & \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})} & \cdots & \mathbf{0} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \hline \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{0} & \mathbf{2} & \cdots & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{2} & \mathbf{0} & \cdots & \mathbf{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{2} & \mathbf{2} & \cdots & \mathbf{0} \end{array}\right]$$

$$= \begin{bmatrix} 2(n-2)^{(n-2)}(J_{n-1} - I_{n-1}) & J_{(n-1)\times n} \\ J_{n\times(n-1)} & 2(J_n - I_n) \end{bmatrix}$$

$$= \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.$$

In the current case, $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}})$ is divided into four blocks, where the first block is $\boldsymbol{T}_1$, which is a block of $(\boldsymbol{n}-\mathbf{1})\times(\boldsymbol{n}-\mathbf{1})$ matrix with zero diagonal and all non-diagonal entries as $\mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})}$. In the next two blocks, we have $\boldsymbol{T}_2$ and $\boldsymbol{T}_3$ matrices, which are of the size $(\boldsymbol{n}-\mathbf{1})\times\boldsymbol{n}$ and $\boldsymbol{n}\times(\boldsymbol{n}-\mathbf{1})$, respectively, whose all entries are equal to one. The last block is $\boldsymbol{T}_4$, which is an $\boldsymbol{n}\times\boldsymbol{n}$ matrix with zero diagonal, and all non-diagonal entries are equal to two. Then, we obtain the characteristic polynomial of $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}})$ from the following determinant

$$\boldsymbol{P}_{\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}})}(\boldsymbol{\lambda}) = \left|\boldsymbol{\lambda I}_{\mathbf{2}\boldsymbol{n}-\mathbf{1}} - \boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}})\right|$$
$$= \begin{vmatrix} (\boldsymbol{\lambda} + \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})})\boldsymbol{I}_{\boldsymbol{n}-\mathbf{1}} - \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})}\boldsymbol{J}_{\boldsymbol{n}-\mathbf{1}} & -\boldsymbol{J}_{(\boldsymbol{n}-\mathbf{1})\times\boldsymbol{n}} \\ -\boldsymbol{J}_{\boldsymbol{n}\times(\boldsymbol{n}-\mathbf{1})} & (\boldsymbol{\lambda}+\mathbf{2})\boldsymbol{I}_{\boldsymbol{n}} - \mathbf{2}\boldsymbol{J}_{\boldsymbol{n}} \end{vmatrix}$$
.

By using Lemma 2.1, with $\boldsymbol{a} = \mathbf{2}(\boldsymbol{n}-\mathbf{2})^{(\boldsymbol{n}-\mathbf{2})}$, $\boldsymbol{b} = \mathbf{2}$, $\boldsymbol{c} = \mathbf{1}$, $\boldsymbol{d} = \mathbf{1}$, $\boldsymbol{n}_1 = \boldsymbol{n}-\mathbf{1}$ and $\boldsymbol{n}_2 = \boldsymbol{n}$, we get the required result.

2. When $\boldsymbol{n}$ is even, using Theorem 3.1, we know that $\boldsymbol{d}_{\boldsymbol{a}^i} = \boldsymbol{n}-\mathbf{3}$ and $\boldsymbol{d}_{\boldsymbol{a}^i\boldsymbol{b}} = \mathbf{1}$, for all $\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}$. Then, using the fact that $\boldsymbol{Z}(\boldsymbol{D}_{2n}) = \{\boldsymbol{e}, \boldsymbol{a}^{\frac{n}{2}}\}$, we have $\mathbf{2}\boldsymbol{n}-\mathbf{2}$ vertices in $\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}}$. The set of vertices consists of $\boldsymbol{n}-\mathbf{2}$ vertices of the form $\boldsymbol{a}^{\boldsymbol{i}}$, with $\boldsymbol{i} \ne \boldsymbol{n}, \frac{\boldsymbol{n}}{\mathbf{2}}$ and $\boldsymbol{n}$ vertices of the form $\boldsymbol{a}^{\boldsymbol{i}}\boldsymbol{b}$, for $\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}$. Correspondingly, the DES matrix for $\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}}$ is a $(\mathbf{2}\boldsymbol{n}-\mathbf{2})\times(\mathbf{2}\boldsymbol{n}-\mathbf{2})$ matrix, $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}}) = [\boldsymbol{des}_{\boldsymbol{pq}}]$ whose entries are:
   (i) $\boldsymbol{des}_{\boldsymbol{pq}} = (\boldsymbol{n}-\mathbf{3})^{\boldsymbol{n}-\mathbf{3}} + (\boldsymbol{n}-\mathbf{3})^{\boldsymbol{n}-\mathbf{3}} = \mathbf{2}(\boldsymbol{n}-\mathbf{3})^{\boldsymbol{n}-\mathbf{3}}$, for $\boldsymbol{p} \ne \boldsymbol{q}$, and $\mathbf{1} \le \boldsymbol{p},\boldsymbol{q} \le \boldsymbol{n}-\mathbf{2}$,
   (ii) $\boldsymbol{des}_{\boldsymbol{pq}} = (\boldsymbol{n}-\mathbf{3})^{\mathbf{1}} + (\mathbf{1})^{\boldsymbol{n}-\mathbf{3}} = \boldsymbol{n}-\mathbf{2}$, for $\mathbf{1} \le \boldsymbol{p} \le \boldsymbol{n}-\mathbf{2}$ and $\boldsymbol{n}-\mathbf{1} \le \boldsymbol{q} \le \mathbf{2}\boldsymbol{n}-\mathbf{2}$,
   (iii) $\boldsymbol{des}_{\boldsymbol{pq}} = (\mathbf{1})^{\boldsymbol{n}-\mathbf{3}} + (\boldsymbol{n}-\mathbf{3})^{\mathbf{1}} = \boldsymbol{n}-\mathbf{2}$, for $\boldsymbol{n}-\mathbf{1} \le \boldsymbol{p} \le \mathbf{2}\boldsymbol{n}-\mathbf{2}$ and $\mathbf{1} \le \boldsymbol{q} \le \boldsymbol{n}-\mathbf{2}$,
   (iv) $\boldsymbol{des}_{\boldsymbol{pq}} = (\mathbf{1})^{\mathbf{1}} + (\mathbf{1})^{\mathbf{1}} = \mathbf{2}$, for $\boldsymbol{p} \ne \boldsymbol{q}$, $\boldsymbol{n}-\mathbf{1} \le \boldsymbol{p} \le \mathbf{2}\boldsymbol{n}-\mathbf{2}$ and $\boldsymbol{n}-\mathbf{1} \le \boldsymbol{q} \le \mathbf{2}\boldsymbol{n}-\mathbf{2}$,
   (v) $\boldsymbol{des}_{\boldsymbol{pq}} = \mathbf{0}$, for $\boldsymbol{p} = \boldsymbol{q}$.

We can construct $\boldsymbol{DES}(\boldsymbol{\Gamma}_{\boldsymbol{D}_{2n}})$ as the following:

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$$\begin{bmatrix} 0 & 2(n-3)^{(n-3)} & \cdots & 2(n-3)^{(n-3)} & n-2 & n-2 & \cdots & n-2 \\ 2(n-3)^{(n-3)} & 0 & \cdots & 2(n-3)^{(n-3)} & n-2 & n-2 & \cdots & n-2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-3)^{(n-3)} & 2(n-3)^{(n-3)} & \cdots & 0 & n-2 & n-2 & \cdots & n-2 \\ n-2 & n-2 & \cdots & n-2 & 0 & 2 & \cdots & 2 \\ n-2 & n-2 & \cdots & n-2 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2 & n-2 & \cdots & n-2 & 2 & 2 & \cdots & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2(n-3)^{(n-3)}(J_{n-2} - I_{n-2}) & (n-2)J_{(n-2)\times n} \\ (n-2)J_{n\times(n-2)} & 2(J_n - I_n) \end{bmatrix}$$
$$= \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}.$$

In the current case, $DES(\Gamma_{D_{2n}})$ is divided into four blocks, where the first block we have U_1 which is a block of $(n-2)\times(n-2)$ matrix with zero diagonal and all non-diagonal entries as $2(n-3)^{(n-3)}$. The next two blocks are U_2 and U_3, which are of the size $(n-2)\times n$ and $n\times(n-2)$, respectively, whose all entries are equal to $n-2$. The last block is U_4, which is an $n\times n$ matrix with zero diagonal, and all non-diagonal entries are equal to two. Then, we obtain the characteristic polynomial of $DES(\Gamma_{D_{2n}})$ from the following determinant

$$P_{DES(\Gamma_{D_{2n}})}(\lambda) = |\lambda I_{2n-2} - DES(\Gamma_{D_{2n}})|$$
$$= \begin{vmatrix} (\lambda + 2(n-3)^{(n-3)})I_{n-2} - 2(n-3)^{(n-3)}J_{n-2} & -(n-2)J_{(n-2)\times n} \\ -(n-2)J_{n\times(n-2)} & (\lambda+2)I_n - 2J_n \end{vmatrix}.$$

By using Lemma 2.1, with $a = 2(n-3)^{(n-3)}$, $b = 2$, $c = n-2$, $d = n-2$, $n_1 = n-2$ and $n_2 = n$, we obtain the result.

The illustration of the above theorem is given by the following examples for $n = 4$ and $n = 5$.

Example 5. In Example 1, we obtained the commuting graph of D_8. Since the degree of each vertex is one, then we will have a 6×6 DES matrix of Γ_{D_8} as follows:

$$DES(\Gamma_{D_8}) = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

Hence, the characteristic polynomial of $DES(\Gamma_{D_8})$ is $P_{DES(\Gamma_{D_8})}(\lambda) = \det(\lambda I_6 - DES(\Gamma_{D_8})) = (\lambda+2)(\lambda+2)^3(\lambda^2 - 8\lambda - 20) = (\lambda+2)^5(\lambda - 10)$. Using Maple™, we confirmed that the eigenvalues of $DES(\Gamma_{D_8})$ are $\lambda = -2$ with multiplicity 5 and a single $\lambda = 10$. Therefore, $E_{DES}(\Gamma_{D_8}) = 5|-2| + |10| = 20$.

Example 6. In Example 2, we have presented the commuting graph of D_{10}. Then, we have a 9×9 DES matrix of $\Gamma_{D_{10}}$ as follows:

$$DES(\Gamma_{D_{10}}) = \begin{bmatrix} 0 & 54 & 54 & 54 & 1 & 1 & 1 & 1 & 1 \\ 54 & 0 & 54 & 54 & 1 & 1 & 1 & 1 & 1 \\ 54 & 54 & 0 & 54 & 1 & 1 & 1 & 1 & 1 \\ 54 & 54 & 54 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

Hence, the characteristic polynomial of $DES(\Gamma_{D_{10}})$ is $P_{DES(\Gamma_{D_{10}})}(\lambda) = \det(\lambda I_9 - DES(\Gamma_{D_{10}})) = (\lambda + 54)^3(\lambda+2)^4(\lambda^2 - 170\lambda + 1276)$. Using Maple™, we confirmed that the eigenvalues of $DES(\Gamma_{D_{10}})$ are $\lambda = -54$ with multiplicity 3, $\lambda = -2$ with multiplicity 4 and $\lambda = 85 \pm 3\sqrt{661}$. Thus, $E_{DES}(\Gamma_{D_{10}}) = 3|-54| + 4|-2| + |85 + 3\sqrt{661}| + |85 - 3\sqrt{661}| = 340$.

Theorem 3.5: Let $\Gamma_{D_{2n}}$ be the commuting graph of D_{2n}. Then

1. for the odd n,
$$E_{DES}(\Gamma_{D_{2n}}) = 4(n-2)^{n-1} + 4(n-1),$$
2. and for the even n,
$$E_{DES}(\Gamma_{D_{2n}}) = \begin{cases} 20, & \text{if } n = 4 \\ 4(n-3)^{n-2} + 4(n-1), & \text{if } n > 4 \end{cases}.$$

Proof.

1. By Theorem 3.4 (1) for the odd n, the characteristic polynomial of $DES(\Gamma_{D_{2n}})$ has four eigenvalues, with the first eigenvalue is $\lambda_1 = -2(n-2)^{n-2}$ of multiplicity $n-2$, and the second eigenvalue is $\lambda_2 = -2$ of multiplicity $n-1$. The quadratic formula gives the other two eigenvalues, which are $\lambda_3, \lambda_4 = (n-2)^{n-1} + (n-1) \pm \sqrt{((n-2)^{n-1} - (n-1))^2 + n(n-1)}$, and both of them are positive real numbers. Hence, the DES energy for $\Gamma_{D_{2n}}$ is
$$E_{DES}(\Gamma_{D_{2n}}) = (n-2)|-2(n-2)^{n-2}| + (n-1)|-2| + \left|(n-2)^{n-1} + (n-1) \pm \sqrt{((n-2)^{n-1} - (n-1))^2 + n(n-1)}\right|$$
$$= 2(n-2)^{n-1} + 2(n-1) + 2(n-2)^{n-1} + 2(n-1)$$
$$= 4(n-2)^{n-1} + 4(n-1).$$

2. By Theorem 3.4 (2) for the even n, the characteristic polynomial of $DES(\Gamma_{D_{2n}})$ has four eigenvalues, with the first eigenvalue is $\lambda_1 = -2(n-3)^{n-3}$ of multiplicity $n-3$, and the second eigenvalue is $\lambda_2 = -2$ of multiplicity $n-1$. The quadratic formula gives the other two eigenvalues, which leads to two cases. First, when $n = 4$, they are a positive real number, and the other is negative. It is evident from Example 5 that $E_{DES}(\Gamma_{D_{2n}}) = 20$. Meanwhile, for $n > 4$, the last two eigenvalues are positive real numbers given by $\lambda_3, \lambda_4 = (n-3)^{n-2} +$

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$$(n-1) \pm \sqrt{\left((n-3)^{n-2}-(n-1)\right)^2+n(n-2)^3}.$$

Thus, the DES energy for $\Gamma_{D_{2n}}$ is

$$E_{DES}(\Gamma_{D_{2n}}) = (n-3)|-2(n-3)^{n-3}| + (n-1)|-2| + \Bigg|(n-3)^{n-2}+(n-1) \pm \sqrt{\left((n-3)^{n-2}-(n-1)\right)^2+n(n-2)^3}\Bigg|$$

$$= 4(n-3)^{n-2}+4(n-1).$$

4. Conclusion

This paper has given the general formula of degree exponent sum (DES) energy of commuting graphs for dihedral groups. In particular, $E_{DES}(\Gamma_{D_{2n}}) = 4(n-2)^{n-1} + 4(n-1)$ when n is odd. On the other hand, there are two cases for n is even, namely $E_{DES}(\Gamma_{D_{2n}}) = 20$ if $n = 4$ and $E_{DES}(\Gamma_{D_{2n}}) = 4(n-3)^{n-2} + 4(n-1)$ if $n > 4$. This happens as a result of the difference between the quadratic polynomial roots, which is a part of the corresponding characteristic polynomial of $DES(\Gamma_{D_{2n}})$.

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